MONOMIAL GROUPS

by

Ralph B. Crouch
B.S., Murray State College, 1942
M.S., University of Illinois, 1947

Submitted to the Department of Mathematics and the Faculty of the Graduate School of the University of Kansas in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

Advisory Committee:

Redacted Signature

Redacted Signature

Redacted Signature

July, 1954

ACKNOWLEDGEMENT

I wish to express my deep appreciation to Professor W. R. Scott for encouraging guidance and helpful suggestions during the course of this work.

Ralph Crouch

Lawrence, Kansas

July, 1954

TABLE OF CONTENTS

INTRODUCTION

Let U be a set with n elements, where n is a finite cardinal. Let H be a fixed group. A <u>monomial substitution</u> y is a transformation that maps every x of the set U in a one-to-one fashion into an x of U multiplied by an element h_x of H. If an operation is defined between monomial substitutions by successive application, the set of all monomial substitutions over H is a group which we denote by Σ_n. Ore [1]* has called this the <u>complete monomial group</u> over H or the <u>symmetry</u> over H.

Ore [1] has shown that the subset of elements of the symmetry Σ_n that map each x of U onto an element of H multiplied by the same x form a normal subgroup of Σ_n, <u>the basis group</u>. We shall denote the basis group by V. The subset of elements of Σ_n that map each x of U onto an x of U multiplied by the identity of H form a subgroup, which we will denote by S, that is isomorphic to the symmetric group on the set U. He has further established that $\Sigma_n = V \cup S$, $V \cap S = E$, where E is the identity of Σ_n. That is, the symmetry <u>splits</u> over the basis group. Ore [1] has presented a complete solution to the problem of finding all representative groups in this splitting. Another result obtained was the determination of all normal subgroups of Σ_n. All of the

---

* Numbers in square brackets refer to bibliography.

automorphisms of Σ_n were also obtained. The investigation was concluded with the study of imbedding an arbitrary group in a monomial group.

This paper generalizes the monomial group by removing the requirement that U be a finite set. Furthermore, the group H is arbitrary throughout the entire thesis. If $o(U) = B = \aleph_u$, $u \geq 0$, where $o(U)$ means the number of elements of U, then a monomial substitution over an arbitrary fixed group H is defined as for the case where $o(U) = n < \aleph_0$. With an operation between monomial substitutions again defined as successive applications, the set of all monomial substitutions over H form a group Σ_B.

The splitting of Σ_B over the basis group is discussed and a complete solution for the determination of all representative groups in the most general case has been found. Corresponding theorems for various subgroups of Σ_B are also found. All of the normal subgroups of various subgroups of the symmetry have been determined. Some progress toward the determination of the automorphisms of the general monomial group has been made by showing that the basis group is characteristic for some subgroups of Σ_B.

In addition, the subgroup $\Sigma_{n,A}$ of Σ_n that has elements which can be written as the product of elements of the basis group multiplied by elements from the alterating group on U is discussed. The problem of describing all representative groups in the splitting over the basis group is completely solved. All of the normal subgroups for $n \geq 5$, $n = 2$ are determined.

Since Σ_B splits over the basis group with a group isomorphic to the infinite symmetric group on U, it is necessary to give some theorems and proofs about the infinite symmetric and alterating groups essential for later investigation of Σ_B. This is done in Chapter I.

In the second chapter the more elementary topics such as transformation, center, centralizer, etc. are discussed. The center of the symmetry is found; a normal form for elements of the symmetry is determined; and the centralizer of any element of the symmetry is found.

Perhaps the most difficult problem in this paper is the determination in Chapter III of all representative groups for the splitting over the basis group. Necessary and sufficient conditions for the symmetry to split regularly are found. In addition, the splitting of $\Sigma_{n,A}$ over the basis group is discussed, and a complete solution for constructing representative groups is given. For this group also, necessary and sufficient conditions for the group to split regularly are given.

In Chapter IV all of the normal subgroups of various subgroups of Σ_B are determined. In one case the normal subgroups are less complicated than for the case where $o(U) = n$. All of the normal subgroups of $\Sigma_{n,A}$ are determined for $n = 2$, $n \geq 5$.

The final chapter is devoted to showing that the basis group is a characteristic subgroup for some of the subgroups of Σ_B. It is also shown that the basis group is a characteristic subgroup for $\Sigma_{n,A}$.

The paper leaves unanswered some questions corresponding to known results when $o(U) = n$. However, the question of the splitting of the symmetry over the basis group would now seem to be completely solved. This is true not only for the case U is infinite but also for the group $\Sigma_{n,A}$, for all n.

The normal subgroups of Σ_B are undetermined in the most general case. The normal subgroups of $\Sigma_{n,A}$ for $n = 3, 4$ are also undetermined. When $n \geq 5$ this problem is completely solved.

The problem of finding all automorphisms of Σ_B is unsolved. It would appear that the determination of all normal subgroups of Σ_B would be a necessary preliminary step toward answering this question. All of the groundwork for finding the automorphisms of $\Sigma_{n,A}$ seems to have been finished.

CHAPTER I

1. Introduction

In this chapter we give that portion of the theory of infinite symmetric and alternating groups which is pertinent to the theory of monomial groups developed later.

For the results of this chapter I am indebted to Professor W. R. Scott [2] who has made available to me in manuscript form these results. The proof presented here of Theorem 10 and the proof of Theorem 11 are due to Professor Scott. The proof of Theorem 7 is due to Schreier and Ulam [4]. The remaining proofs (and Theorem 10) are due to Baer [5].

2. Definitions and Preliminaries

Let d be the cardinal of the set of integers. Let B be an infinite cardinal, B^+ the successor of B, U a set such that $o(U) = B$, where $o(U)$ denotes the number of elements in U, and let C be such that $d \leqq C \leqq B^+$. Let s be a one-to-one transformation of U onto itself and let $U(s)$ be the set of x belonging to U such that s moves x. Denote by $S(U,C)$ the set of s such that the number of elements x of U that s moves is less than C. The product of two transformations s and s_0 in $S(U,C)$ is defined to be that transformation resulting from successive application of s and s_0 in the given order. Then we have:

<u>Theorem 1</u>: $S(U,C)$ is a group.

<u>Theorem 2</u>: If U and U_0 have the same number of elements, then $S(U,C) \cong S(U_0,C)$.

Theorem 2 justifies the use of the notation $S(B,C)$ for $S(U,C)$ although the notation $S(U,C)$ will still be used when necessary. The groups $S(B,C)$ are called the <u>infinite symmetric groups</u>. We shall use I for the identity of the groups.

If $o(U(s))$ is finite then s may be considered as an element of the finite symmetric group on those objects. Let $A(U,d)$ be the subset of $S(U,d)$ consisting of those elements s which are in the alternating group $A(U(s))$ of $U(s)$.

Theorem 3: $A(U,d)$ is a group.

Theorem 4: If U and U_0 have the same number of elements, then $A(U,d) \cong A(U_0,d)$.

Theorem 4 justifies the use of the notation $A(B,d)$ for $A(U,d)$. The groups $A(B,d)$ are called the <u>infinite alternating groups</u>.

Let s belong to $S(B,C)$. Then the minimal non-empty subsets W of U such that $Ws = W$ are called <u>cycles</u>. If x_1 belongs to U, and if n is the minimum positive integer such that $x_1 s^n = x_1$, then the transformation

$$c = \begin{pmatrix} x_1, & x_2, & \ldots, & x_{n-1}, & x_n \\ x_2, & x_3, & \ldots, & x_n, & x_1 \end{pmatrix} = (x_1, x_2, \ldots, x_n)$$

is a cycle of s. If $x_1 s^n \neq x_1$ for all $n > 0$, then the transformation

$$c = \begin{pmatrix} \ldots, & x_{-2}, & x_{-1}, & x_0, & x_1, & x_2, & \ldots \\ \ldots, & x_{-1}, & x_0, & x_1, & x_2, & x_3, & \ldots \end{pmatrix}$$

$$= (\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots)$$

is a cycle, and all of the x_i, $i = 0, 1, -1, \ldots$, are distinct. A cycle with n distinct x's is called an n-cycle; $n = 1, 2, \ldots; d$.

For each x of U and a given s it is clear that x appears in one and only one cycle of s. Hence s determines its set of cycles. Conversely, a set of disjoint cycles which together contain all x of U determine an element s of $S(B,B^+)$. It is customary to say that s is the product of its cycles, and to use corresponding notation. Since this product may be infinite, however, this procedure must be taken with a grain of salt.

We shall denote by $k_n(s)$ the number of n-cycles in s; $n = 1, 2, \ldots; d$.

<u>Theorem 5</u>: Let s, $s_\circ$ be elements of $S(B,B^+)$. Then the cycles of $s^{-1}s_\circ s$ are obtained from the cycles $(x_1, \ldots, x_n)$ or $(\ldots, x_{-1}, x_0, x_1, \ldots)$ of $s_\circ$ by replacing x_i by the element x_{j_i} that s maps x_i onto.

Proof:

$s^{-1}s_\circ s =$

$$\begin{pmatrix} \ldots, & x_{j_i}, & \ldots \\ \ldots, & x_i, & \ldots \end{pmatrix} \begin{pmatrix} \ldots, & x_i, & \ldots \\ \ldots, & x_{i+1}, & \ldots \end{pmatrix} \begin{pmatrix} \ldots, & x_i, & x_{i+1}, & \ldots \\ \ldots, & x_{j_i}, & x_{j_{i+1}}, & \ldots \end{pmatrix}$$

$$= \begin{pmatrix} \ldots, & x_{j_i}, & \ldots \\ \ldots, & x_{j_{i+1}}, & \ldots \end{pmatrix}$$

where $x_{j_{i+1}} = x_{j_1}$ if $i = n$ of a finite cycle.

Theorem 6: If G is an infinite symmetric or alternating group, then s is conjugate to $s_\circ$ within G if and only if $k_n(s) = k_n(s_\circ)$ for all n.

Proof: If $y^{-1}sy = s_\circ$ then it follows from Theorem 5 that $k_n(s) = k_n(s_\circ)$.

Conversely, let us assume that $k_n(s) = k_n(s_\circ)$ for all n. This means that there is a one-to-one mapping between cycles of s and cycles of $s_\circ$ which preserves length. This mapping may be used to construct a one-to-one mapping of U onto itself such that $s = y^{-1}s_\circ y$ using Theorem 5. If $G = S(B,B^+)$ this completes the proof. Assume that $G = S(B,C)$ with $C \leqq B$. Then there exists a set $P \subset U$ such that (i) $(U(s) \cup U(s_\circ)) \subseteq P$, (ii) $o(P-U(s)) = o(P-U(s_\circ))$, and (iii) $o(P) < C$. Then construct s as before for cycles in s of length greater than 1. This maps U(s) onto $U(s_\circ)$. Next map P-U(s) onto $P-U(s_\circ)$ in a one-to-one fashion, and finally map the remainder of U by the identity map. If this compound map is again called y, then $y^{-1}sy = s_o$, and since $U(y) \subseteq P$, y belongs to $S(B,C)$. If $G = A(B,d)$ then s and s_o are in $S(B,d)$ also, so y may be chosen in $S(B,d)$. If y does not belong to $A(B,d)$ and x_α, x_β belong to $U-U(y)$, then $y(x_\alpha, x_\beta) = y_o$ belongs to $A(B,d)$ and $y_o^{-1}sy_o = y$.

<u>Theorem 7</u>: Let G be an infinite symmetric or alternating group, and let y belong to G. Then there exist elements s and s_o in G such that $y = ss_o$ and $s^2 = s_o^2 = I$.

<u>Proof</u>: Note the formulas:

$$(x_1, \ldots, x_{2n}) = \{(x_n, x_{n+1})(x_{n-1}, x_{n+2})\ldots(x_1, x_{2n})\}\{(x_{n+1})(x_n, x_{n+2})\ldots(x_2, x_{2n})\}$$

$$(x_1, \ldots, x_{2n+1}) = \{(x_n, x_{n+1})(x_{n-1}, x_{n+2})\ldots(x_1, x_{2n})\}\{(x_{n+1})(x_n, x_{n+2})\ldots(x_1, x_{2n+1})\}$$

$$(\ldots, x_{-1}, x_0, x_1, \ldots) = \{(x_0, x_{-1})(x_1, x_{-2})\ldots(x_n, x_{-n-1})\ldots\}\{(x_0)(x_1, x_{-1})\ldots(x_n, x_{-n})\ldots\}.$$

Decompose y into cycles $y = \prod_\alpha c_\alpha$. By the above formulas, $c_\alpha = D_\alpha E_\alpha$ where $D_\alpha^2 = I = E_\alpha^2$, and D_α and E_α move no letters unmoved by c_α (if c_α is a 1-cycle let $D_\alpha = E_\alpha = I$). Let $s = \prod_\alpha D_\alpha$, $s_o = \prod_\alpha E_\alpha$. Thus $y = ss_o$, $s^2 = I = s_o^2$. Furthermore, s and s_o are in G except possibly in the case $G = A(B,d)$. In this case if s does not belong to

$A(B,d)$, then s_0 does not belong to $A(B,d)$, and since s and s_0 are finite permutations, there exists x_α and x_β in $U-U(s)-U(s_0)$. Let $s' = s(x_\alpha, x_\beta)$, $s_0' = s_0(x_\alpha, x_\beta)$. Thus $y = s's_0'$, s' and s_0' are in G, and $(s')^2 = I = (s_0')^2$.

3. Normal Subgroups

It will now be shown that the only normal subgroups of the infinite symmetric and alternating groups are again infinite symmetric and alternating groups.

Theorem 8: $A(B,d)$ is simple.

Proof: Assume that N is a normal subgroup of $A(B,d)$, $N \neq I$, and let s be any element of $A(B,d)$, s_0 an element of N with $s_0 \neq I$. Let U_0 be a finite set of order>4 such that $\{U(s) \cup U(s_0)\} \subseteq U_0$. Then s and s_0 may be considered as elements of the finite alternating group $A(U_0, n+1)$ where $n = o(U_0)$. Since N is normal in $A(B,d)$, $N \cap A(U_0, n+1)$ is normal in $A(U_0, n+1)$. Hence by the simplicity of the finite alternating groups of degree greater than 4, $A(U_0, n+1) \subseteq N$. Thus s belongs to N, and since s was arbitrary in $A(B,d)$ we have $N = A(B,d)$.

Theorem 9: $S(B,d)$ contains just one proper normal subgroup, $A(B,d)$.

Proof: $A(B,d)$ is clearly of index 2 in $S(B,d)$, hence a proper normal subgroup. Assume that N is a proper normal subgroup of $S(B,d)$ with $N \neq A(B,d)$. Let $R = N \cap A(B,d)$. Then R is a normal subgroup of $A(B,d)$

and $R \neq A(B,d)$. It follows from Theorem 8 that $R = I$. By the second isomorphism theorem, $o(N) = 2$, but such an N is clearly not normal by Theorem 6.

Theorem 10: The proper normal subgroups of $S(B,C)$ for $C > d$ are the groups $A(B,d)$ and $S(B,D)$ with $D < C$.

Proof: The fact that $A(B,d)$ and $S(B,D)$ are proper normal subgroups follows from their definitions and Theorem 6.

Conversely, let N be a proper normal subgroup of $S(B,C)$. If N contains only finite permutations, then by Theorem 9, $N = S(B,d)$ or $A(B,d)$. We assert that if there exists an $s_\circ$ belonging to N such that $o(U(s_\circ)) = D$ then $S(B,D^+) \subseteq N$. This assertion is equivalent to the unproved portion of the theorem. The proof of this assertion will be broken up into four steps.

(i) There exists an s belonging to N such that $k_3(s) = D$, $k_1(s) = B$, and $k_n(s) = 0$ if $n \neq 1$ or 3.

If $\sum\limits_{2<n<d} k_n(s_\circ) + dk_d(s_\circ) = D$, then form an element s_1 conjugate to $s_\circ$ in $S(B,C)$ as follows. If $s_\circ$ contains a 1-cycle or a 2-cycle, let s_1 also contain the cycle. Well order the finite cycles of length greater than 2. For every other such cycle in $s_\circ$, $(x_1 \ldots, x_n)$, let s_1 contain the n-cycle $(x_{n-1}, x_n, x_{n-2}, x_{n-3}, \ldots, x_1)$. For the remaining finite cycles of $s_\circ$ let s_1 contain

their inverses. For all d-cycles $(\ldots, x_{-1}, x_0, x_1, \ldots)$ in s. let s_1 contain the d-cycle $(\ldots, x_4, x_3, x_1, x_2, x_0, x_{-1}, x_{-3}, x_{-2}, x_{-4}, \ldots)$. Then $s_\circ s_1$ belongs to N and the following 3-cycles occur in the cyclic decomposition of $s_o s_1$: (x_{n-2}, x_n, x_{n-1}) for every other finite cycle of s. and $\ldots, (x_2, x_1, x_0), (x_{-2}, x_{-3}, x_{-4}), \ldots$ for every d cycle of s. Furthermore, there are no other n-cycles for $n > 1$. Let $s = s_o s_2$. Then $k_3(s) = D$ and since enough 1-cycles have been saved out $k_1(s) = B$. Finally, $k_n(s) = 0$ if $n \neq 1$ or 3.

If $\sum_{2<n<d} k_n(s_\circ) + dk_d(s_\circ) < D$, then $k_2(s_\circ) = D$. For every three 2-cycles in s., (x_1, x_2), (x_3, x_4) and (x_5, x_6), let s_1 contain the 2-cycles $(x_1, x_3), (x_2, x_5)$ and (x_4, x_6). Then $s_\circ s_1$ contains the 3-cycles (x_1, x_5, x_4) and (x_2, x_3, x_6). Hence $k_3(s_\circ s_1) = D$ and since s_1 clearly can be chosen conjugate to s., consistent with the above requirements, the first case applies. This shows that the assertion in (i) is true in any case.

(ii) There exists an s belonging to N such that $k_2(s) = D$, $k_1(s) = B$, $k_n(s) = 0$ if $n \neq 1$ or 2.

For by (i) there exist elements s. and s_1 in N such that the cyclic decomposition of each element consists of D sets of one 3-cycle and one 1-cycle, together with B common additional 1-cycles. Further require that if a typical combination of one 3-cycle and one 1-cycle in $s_\circ$ is (x_1, x_2, x_3) and (x_4), then s_1 contains the combination (x_1, x_2, x_4) and (x_3). If $s = s_\circ s_1$, then s contains D combinations of the type (x_1, x_4) and (x_2, x_3) and B additional 1-cycles. Therefore, s has the required properties.

(iii) N contains every element of order 2 in $S(B,D^+)$.

Let $F < D$ (F may be finite or 0). Let s be of the type described in (ii), and let $s_\circ$ contain all but F of the 2-cycles in s, and no other n-cycles for $n > 1$. Then $s_\circ$ belongs to N, and $k_2(ss_\circ) = F$, $k_1(ss_\circ) = B$, and $k_n(ss_\circ) = 0$ otherwise. Unless $D = B$ we are finished, by Theorem 6 and (ii).

If $D = B$, let $F < D$ (again F may be finite or 0). Let s be of the type described in (ii), and let $s_\circ$ also be of that type, and such that $U(s) \cap U(s_\circ)$ is empty, while $o(U-U(s)-U(s_\circ)) = F$. Then $k_2(ss_\circ) = D$, $k_1(ss_\circ) = F$, and $k_n(ss_\circ) = 0$ otherwise. Since any element s_1 of order 2 has $k_n(s_1) = 0$ for $n > 2$, this shows that all elements of order 2 in $S(B,D^+)$ are contained in N.

(iv) $S(B,D^+) \subseteq N$.

This follows from (iii) and Theorem 7.

<u>Theorem 11</u>: If $C > d$ and if N is a proper normal subgroup of $S(B,C)$, then there exists a subgroup K of $S(B,C)/N$ such that $K \cong S(B,C)$.

<u>Proof</u>: By Theorem 10 there exists a cardinal $D < C$ such that $N \subseteq S(B,D)$. Let U be the union of D disjoint sets U_α, each of cardinal B. Let θ_α be an isomorphism of $S(U,C)$ onto $S(U_\alpha,C)$ and let $s\theta = \prod_\alpha s\theta_\alpha$ for s belonging to $S(U,C)$. It is clear that θ is defined and $s = s_o$ implies that $s\theta = s_o\theta$. If $\prod_\alpha s\theta_\alpha = \prod_\alpha s_o\theta_\alpha$, then $s\theta_\alpha = s_o\theta_\alpha$ for all α since the U_α were disjoint. It follows that $s = s_o$. The fact that the U_α are disjoint is also a sufficient condition to show that the multiplication is preserved by the correspondence. Furthermore, $o(U(s\theta)) = D \cdot o(U(s))$. It follows that θ is an isomorphism of $S(U,C)$ onto a subgroup L of $S(U,C)$. Since $o(U(s\theta)) \geqq D$ if $s \neq I$, we have $L \cap N = I$. Letting $K = LN/N$ the theorem follows.

CHAPTER II

The Symmetries

1. Definitions

Let H be some group, finite or infinite. We will write e for the identity of H. Let S be a set of order $B = \aleph_u$ for $u \geq o$. We shall denote the elements of S by $\{x\}$. The set may be well ordered so that we may write $x_1, x_2, \ldots; x_\omega, \ldots$.

Definition: <u>A monomial substitution over H is a linear transformation where each variable x of S is mapped by a one-to-one mapping onto some other variable multiplied by an element of H.</u>

A substitution y will be written

$$(1) \quad y = \begin{pmatrix} x_1 \,, & x_2 \,, & \ldots, & x_\varepsilon \,, & \ldots \\ h_1 x_{i_1}, & h_2 x_{i_2}, & \ldots, & h_\varepsilon x_{i_\varepsilon}, & \ldots \end{pmatrix}$$

where ε is some ordinal. The h_ε will be called <u>factors</u> of y.

The multiplication $h_\varepsilon x_{i_\varepsilon}$ is a formal one to be taken only as a pair $(h_\varepsilon, x_{i_\varepsilon})$ with the associative property $h(kx) = (hk)x$.

Definition: <u>If y is given by (1) and y_1 is given by</u>

$$y_1 = \begin{pmatrix} x_1 \,, & x_2 \,, & \ldots, & x_\varepsilon \,, & \ldots \\ k_1 x_{j_1}, & k_2 x_{j_2}, & \ldots, & k_\varepsilon x_{j_\varepsilon}, & \ldots \end{pmatrix}$$

<u>then the product yy_1 is defined by</u>

$$yy_1 = \begin{pmatrix} x_1 & , & x_2 & ,\ldots, & x_\varepsilon & ,\ldots \\ h_1k_{i_1}x^1_{j_{i_1}} & , & h_2k_{i_2}x^2_{j_{i_2}} & ,\ldots, & h_\varepsilon k_{i_\varepsilon}x^\varepsilon_{j_{i_\varepsilon}} & ,\ldots \end{pmatrix}.$$

By this definition of multiplication the set of monomial substitutions is a group that will be denoted by $\Sigma(H;\ B,\ B^+,\ B^+)$ and called the <u>monomial group</u> of H of degree B or more simply the <u>symmetry</u> of H. The reason for the complexity of the notation for the monomial group is to provide an adequate notation for various subgroups to be discussed later. The identity of the symmetry will be denoted by E. The inverse of y is

$$(2)\quad y^{-1} = \begin{pmatrix} x_{i_1} & , & x_{i_2} & , & \ldots, & x_{i_\varepsilon} & , & \ldots \\ h_1^{-1}x_1 & , & h_2^{-1}x_2 & , & \ldots, & h_\varepsilon^{-1}x_\varepsilon & , & \ldots \end{pmatrix}.$$

If H consists only of the identity element, then $\Sigma(H;\ B,\ B^+,\ B^+)$ is the symmetric group on a set of elements of order B.

<u>Definition: A permutation in $\Sigma(H;\ B,\ B^+,\ B^+)$ is a substitution of the form</u>

$$(3)\quad s = \begin{pmatrix} x_1 & , & x_2 & , & \ldots, & x_\varepsilon & , & \ldots \\ ex_{i_1} & , & ex_{i_2} & , & \ldots, & ex_{i_\varepsilon} & , & \ldots \end{pmatrix}$$

$$= \begin{pmatrix} 1 & , & 2 & , & \ldots, & \varepsilon & , & \ldots \\ i_1 & , & i_2 & , & \ldots, & i_\varepsilon & , & \ldots \end{pmatrix}.$$

The set of permutations form a subgroup of $\Sigma(H;\ B,\ B^+,\ B^+)$ which we will denote, as in Chapter I, by $S(B,B^+)$ and call the <u>permutation subgroup</u> of $\Sigma(H;\ B,\ B^+,\ B^+)$. This subgroup is isomorphic to the ordinary symmetric group on B objects.

Definition: A multiplication in $\Sigma(H; B, B^+, B^+)$ is a substitution which multiplies each variable by a factor in H and hence has the form

$$(4)\quad v = \begin{pmatrix} x_1, & x_2, & \ldots, & x_\varepsilon, & \ldots \\ k_1x_1, & k_2x_2, & \ldots, & k_\varepsilon x_\varepsilon, & \ldots \end{pmatrix}$$

$$= \{k_1, k_2, \ldots, k_\varepsilon, \ldots\}.$$

The set of multiplications form a subgroup of $\Sigma(H; B, B^+, B^+)$ which we will denote by $V(B, B^+)$, and we will call it the basis group of $\Sigma(H; B, B^+, B^+)$.

The basis group is a normal subgroup of the symmetry. For yvy^{-1} as given by (1), (4), and (2) is $\{h_1k_{i_1}h_{i_1}^{-1}, h_2k_{i_2}h_{i_2}^{-1}, \ldots, h_\varepsilon k_{i_\varepsilon}h_{i_\varepsilon}^{-1}, \ldots\}$ which is a multiplication.

The basis group is the strong direct product of groups H_ε^* where H_ε^* is isomorphic to H and H_ε^* consists of the multiplications $v_\varepsilon = \{e, \ldots, e, h_\varepsilon, e, \ldots\}$ with h_ε in the $\varepsilon^{\underline{th}}$ position and h_ε runs through H.

Definition: A scalar in $\Sigma(H; B, B^+, B^+)$ is a multiplication with each factor the same. A scalar is of the form

$$(5)\quad v = \{h, h, \ldots, h, \ldots\} = \{h\}.$$

The scalars are the only elements that commute with permutations. A computation shows that

$$vs = sv = \begin{pmatrix} x_1, & x_2, & \ldots, & x_\varepsilon, & \ldots \\ hx_{i_1}, & hx_{i_2}, & \ldots, & hx_{i_\varepsilon}, & \ldots \end{pmatrix}$$

when v is as in (5) and s as in (3). This shows that scalars commute with all permutations. Now let us assume $ys = sy$ for all s of $S(B, B^+)$ and show that y is a scalar. Let h_α, h_β be factors of y occuring in the $\alpha^{\underline{th}}$ and $\beta^{\underline{th}}$ positions respectively. Then $y(\alpha,\beta)$ has h_α as the $\alpha^{\underline{th}}$ factor and h_β as the $\beta^{\underline{th}}$ factor. But $(\alpha,\beta)y$ has h_β as the $\alpha^{\underline{th}}$ factor and h_α as the $\beta^{\underline{th}}$ factor. Since this is true for all α and β, we see that y must have the same factor in every position. If y sends x_ε into hx_{i_ε} where $i_\varepsilon \neq \varepsilon$, then $y(i_\varepsilon,\alpha)$, where $i_\varepsilon \neq \alpha$, sends x_ε into hx_α. But $(i_\varepsilon,\alpha)y$ sends x_ε into hx_{i_ε} and a contradiction is reached. Therefore, y must send every x into hx and y is a scalar.

The product of two scalars is also a scalar. In fact, the scalars form a subgroup of $\Sigma(H; B, B^+, B^+)$ which is isomorphic to H.

The center of $\Sigma(H; B, B^+, B^+)$ is the set of all scalars $v = \{f, f, \ldots, f, \ldots\} = \{f\}$ where f belongs to the center of H. If y is as in (1) and v as above,

$$yv = \begin{pmatrix} x_1 , & x_2 , & \ldots, & x_\varepsilon , & \ldots \\ h_1 f x_{i_1}^1 , & h_2 f x_{i_2}^2 , & \ldots, & h_\varepsilon f x_{i_\varepsilon}^\varepsilon , & \ldots \end{pmatrix}$$

and

$$vy = \begin{pmatrix} x_1 , & x_2 , & \ldots, & x_\varepsilon , & \ldots \\ f h_1 x_{i_1}^1 , & f h_2 x_{i_2}^2 , & \ldots, & f h_\varepsilon x_{i_\varepsilon}^\varepsilon , & \ldots \end{pmatrix},$$

which are the same since $fh = hf$ for all h of H. This establishes the sufficiency. It has already been shown that an element of the center must be a scalar. It follows from the above computation that a scalar will be in the center only if the factor is in the center of H.

The center of $\Sigma(H; B, B^+, B^+)$ is isomorphic to the center of H, $Z(H) \overset{e}{\cong} Z(\Sigma(H; B, B^+, B^+))$, by $(f)e = \{f\}$.

Any substitution can be factored into a multiplication multiplied by a permutation. For if y is as in (1) we have

$$y = \{h_1, h_2, \ldots, h_\varepsilon, \ldots\} \begin{pmatrix} x_1 & , & x_2 & , & \ldots, & x_\varepsilon & , & \ldots \\ x_{i_1} & , & x_{i_2} & , & \ldots, & x_{i_\varepsilon} & , & \ldots \end{pmatrix}.$$

This shows that

$\Sigma(H; B, B^+, B^+) = V(B,B^+) \cup S(B,B^+)$, $V(B,B^+) \cap S(B,B^+) = E$.

The symmetry $\Sigma(H; B, B^+, B^+)$ has some other subgroups which will be discussed in detail in later work. Let C be some cardinal with $d \leq C \leq B^+$. Then the set of those elements of $\Sigma(H; B, B^+, B^+)$ which can be written $y = vs$ where v is in $V(B, B^+)$ and s is in $S(B,C)$ form a subgroup of $\Sigma(H; B, B^+, B^+)$. We shall denote this subgroup by $\Sigma(H; B, B^+, C)$. It is clear that

$\Sigma(H; B, B^+, C) = V(B,B^+) \cup S(B,C)$, $V(B,B^+) \cap S(B,C) = E$.

The set of elements of $\Sigma(H; B, B^+, B^+)$ which can be written $y = vs$ where v is in $V(B, B^+)$ and s is in $A(B,d)$ form a subgroup of $\Sigma(H; B, B^+, B^+)$ which shall be denoted by $\Sigma_A(H; B, B^+, d)$. It is clear that

$\Sigma_A(H; B, B^+, d) = V(B,B^+) \cup A(B,d)$, $V(B,B^+) \cap A(B,d) = E$.

Another subgroup which is the subject of later investigation is that set of elements of $\Sigma(H;\ B,\ B^+,\ B^+)$ which can be written in the form $y = vs$ where s is in $S(B,d)$ and v has only a finite number of factors different from e. We shall write this group as $\Sigma(H;\ B,\ d,\ d)$. If the subgroup of the basis group whose elements have only a finite number of factors different from e is denoted by $V(B,d)$, we see that

$$\Sigma(H;\ B,\ d,\ d) = V(B,d) \cup S(B,d),\ V(B,d) \cap S(B,d) = E.$$

Finally, the set of those elements of the form $y = vs$ where v belongs to $V(B,d)$ and s belongs to $A(B,d)$ form a group denoted by $\Sigma_A(H;\ B,\ d,\ d)$. Clearly

$$\Sigma_A(H;\ B,\ d,\ d) = V(B,d) \cup A(B,d),\ V(B,d) \cap A(B,d) = E.$$

The above mentioned subgroups are those that are investigated later. However, if B, C, D are infinite cardinals such that $C \leq B^+$, $D \leq B^+$, it is clear that if we define $\Sigma(H;\ B,\ C,\ D) = V(B,C) \cup S(B,D)$, then $V(B,C) \cap S(B,D) = E$.

Now let H be some group, n some natural number, and S a set of order n. Then a monomial substitution over H is of the form $v = \begin{pmatrix} x_1 & , \ldots, & x_n \\ h_1 x_{i_1} & , \ldots, & h_n x_{i_n} \end{pmatrix}$.

The group formed by defining multiplication as before will be denoted by $\Sigma(H;\ n,\ n+1,\ n+1)$. Permutations, multiplications, scalars, permutation group, and basis group can be defined as in the more general case. It follows that $\Sigma(H;\, n,\, n+1,\, n+1) = V(n,\, n+1) \cup S(n,\, n+1)$, $V(n,\, n+1) \cap S(n,\, n+1) = E$. We shall be interested in the subgroup of $\Sigma(H;\, n,\, n+1,\, n+1)$ formed by the set of all $y = vs$ where v belongs to $V(n,\, n+1)$ and s belongs to $A(n,\, n+1)$. We shall denote this group by $\Sigma_A(H;\, n,\, n+1,\, n+1)$ and $\Sigma_A(H;\, n,\, n+1,\, n+1) =$ $V(n,\, n+1) \cup A(n,\, n+1)$, $V(n,\, n+1) \cap A(n,\, n+1) = E$.

2. Cycles and Transformations

In the theory of monomial groups, as in the theory of substitution groups, it is advantageous to introduce cycles of monomial substitutions.

Let y be an arbitrary element of $\Sigma(H;\, B,\, B^+,\, B^+)$. It has been shown that y has a unique decomposition as the product of an element from the basis group multiplied by an element from the permutation group. That is, $y = vs$ where v belongs to $V(B, B^+)$ and s belongs to $S(B, B^+)$. In the previous chapter it was shown that s can be decomposed uniquely into disjoint, commutative cycles of length n where $n = 1, 2, \ldots; d$. For each cycle c_ε of s we can associate a multiplication v_ε which has factors of e in positions corresponding to x which c_ε leaves fixed and factors the same as in v for positions corresponding to those x which c_ε moves. Thus $v_\varepsilon c_\varepsilon$ has one of the two forms

$$v_\varepsilon c_\varepsilon = \begin{pmatrix} x_1, & \ldots, & x_n \\ h_1x_2, & \ldots, & h_nx_1 \end{pmatrix} \quad \text{when } n < d$$

or

$$v_\varepsilon c_\varepsilon = \begin{pmatrix} \ldots, & x_{-1}, & x_0, & x_1, & \ldots \\ \ldots, & h_{-1}x_0, & h_0x_1, & h_1x_2, & \ldots \end{pmatrix} \quad \text{when } n = d.$$

This shows that the unique cyclic decomposition of s determines a decomposition of v. Since the only possible non-identity factors of v_ε are in position occupied by x which c_ε moves, it follows that v_ε commutes with all other cycles c of s as well as with all other multiplications in the decomposition of v. c_ε commutes with all of the multiplications in the decomposition of v except v_ε, so y can be decomposed into the product of $v_\varepsilon c_\varepsilon$ which are disjoint and commutative. It should be remembered that this decomposition may yield an infinite number of the $v_\varepsilon c_\varepsilon$.

Although the above discussion was in terms of elements of $\Sigma(H; B, B^+, B^+)$, it is clear that elements of $\Sigma(H; B, B^+, C)$, where $d \leq C < B^+$, $\Sigma_A(H; B, B^+, d)$, $\Sigma(H; B, d, d)$, $\Sigma_A(H; B, d, d)$, and $\Sigma(H; n, n+1, n+1)$, can be decomposed in a similar fashion since each element of the groups $S(B,C)$, $S(B,d)$, $A(B,d)$, and $S(n, n+1)$ has a unique cyclic decomposition.

Ore [1, p. 19] has investigated the results of transforming a finite cycle of an element of a monomial group. His works show that if c is a cycle of length n and has the form $c = \begin{pmatrix} x_1, & \ldots, & x_n \\ h_1x_2, & \ldots, & h_nx_1 \end{pmatrix}$

that the $n^{\underline{th}}$ power of c is $\{\delta_1, \ldots, \delta_n\}$ where $\delta_1 = h_1 \ldots h_n$, $\delta_2 = h_2 \ldots h_n h_1$, $\ldots$, $\delta_n = h_n h_1 \ldots h_{n-1}$. These elements of H are called the <u>determinants</u> of c. Since $\delta_2 = h_1^{-1}\delta_1 h_1, \ldots, \delta_n = h_{n-1}^{-1}\delta_{n-1}h_{n-1}$, $\delta_1 = h_n^{-1}\delta_n h_n$, there is associated with c a unique determinant class. He then proves that a necessary and sufficient condition for two finite cycles to be conjugate is that they have the same length and the same determinant class. We shall now prove a result for cycles of length d.

<u>Theorem 1</u>: A necessary and sufficient condition that two cycles of length d be conjugate is that they leave the same number of x fixed.

<u>Proof</u>: We shall consider first the case where the cycles are in a monomial group where $B > d$. Then any two cycles c, $c_\circ$ of length d leave the same number of x fixed. We shall show, therefore, that any two such cycles are always conjugate. Let c be given by

$$(6) \quad c = \begin{pmatrix} \ldots, & x_{-1}, & x_0, & x_1, & \ldots \\ \ldots, & h_{-1}x_0, & h_0x_1, & h_1x_2, & \ldots \end{pmatrix},$$

and let $c_\circ$ be given by

$$(7) \quad c_\circ = \begin{pmatrix} \ldots, & x_{i_{-1}}, & x_{i_0}, & x_{i_1}, & \ldots \\ \ldots, & r_{-1}x_{i_0}, & r_0x_{i_1}, & r_1x_{i_2}, & \ldots \end{pmatrix}.$$

Then if we choose an element y of the monomial group that has in its cyclic decomposition

$$(8) \quad c_1 = \begin{pmatrix} \ldots, & x_{-1}, & x_0, & x_1, & \ldots \\ \ldots, & k_{-1}x_{i_{-1}}, & k_0x_{i_0}, & k_1x_{i_1}, & \ldots \end{pmatrix}$$

with k_0 an arbitrary element of H and the remaining k_i satisfying the set of equations

$$
\begin{aligned}
&\cdots\cdots\cdots \\
k_{-2} &= h_{-2}k_{-1}r_{-2}^{-1} \\
k_{-1} &= h_{-1}k_0r_{-1}^{-1} \\
k_1 &= h_0^{-1}k_0r_0 \\
k_2 &= h_1^{-1}k_1r_1 \\
&\cdots\cdots\cdots
\end{aligned}
$$

we see that $y^{-1}cy = c_\circ$ as the following computation shows:

$$(9)\quad y^{-1}cy$$

$$= \begin{pmatrix} \ldots, & x_{i_{-1}}, & x_{i_0}, & x_{i_1}, \ldots \\ \ldots, & k_{-1}^{-1}h_{-1}k_0x_{i_0}, & k_0^{-1}h_0k_1x_{i_1}, & k_1^{-1}h_1k_2x_{i_2}, \ldots \end{pmatrix}$$

$$= c_\circ.$$

Conversely, if c is a cycle of length d given by (6) and if y is an element of the monomial group, then ycy^{-1} moves the same number of x that c moves. For if we write $y = vs$ then $ycy^{-1} = vscs^{-1}v^{-1} = vv_1(scs^{-1})$. By Theorem 6 of Chapter I it follows that scs^{-1} is a cycle of length d so ycy^{-1} leaves B of the x's fixed.

Now let B = d and let c and $c_\circ$ be two cycles of length d. Let m be the number of x unmoved by c and n the number of x unmoved by $c_\circ$. If $n = m = 0$, then we can choose y as before such that $y^{-1}cy = c_\circ$. If $1 \leq n \leq d$, $1 \leq m \leq d$, and $n = m$, we choose y with a cycle c_1 as indicated in (8) with the same restrictions on the factors, and y sends those x that c leaves fixed into those x that c_o leaves fixed with e as a factor. Then again $y^{-1}cy = c_o$. If the number of x that c and c_o leave fixed differs, it will not be possible to find a y such that $y^{-1}cy = c_o$, because $y^{-1}cy$ leaves the same number of x fixed as c.

From the theorem just proved and the corresponding theorem proved by Ore [1, p. 19] about finite cycles, we are able to say that two monomial substitutions y, y_1 of Σ, where $B \geq d$, are conjugate if and only if in their cyclic decomposition the finite cycles can be made to correspond in a one-to-one manner such that corresponding cycles have the same length and determinant class and if the cardinal of the set of infinite cycles is the same for both y and y_1.

Any infinite cycle c as in (6) can be transformed into the normal form

$$c = \begin{pmatrix} \dots, & x_{-1}, & x_0, & x_1, & \dots \\ \dots, & x_0, & x_1, & x_2, & \dots \end{pmatrix}$$

by a proper choice of the factors of c_1 as given by (8). Furthermore, any substitution y is conjugate to a product of cycles without common variables where each cycle is in normal form. We have seen that a transformation of cycles of length d into normal form is possible using a substitution involving only those x which c moves. Ore [1, p. 20] has shown that any finite cycle can be transformed to the normal form

$c = \begin{pmatrix} x_1, & \ldots, & x_{n-1}, & x_n \\ x_2, & \ldots, & x_n, & ax_1 \end{pmatrix}$ where a is any element of the determinant class of c. This transformation involves only those variables which c moves. Therefore, all of the cycles of any substitution may be put in normal form by one transformation.

3. Centralizers

Ore [1, pp. 20, 21] has found the centralizer of a substitution which has only a finite number of finite cycles in its cyclic decomposition. We shall list here the results of his investigation and use them later to find the centralizer of an element of the symmetry $\Sigma(H; B, B^+, B^+)$. Let c_a be a cycle of length n and of the form $c_a = \begin{pmatrix} x_1, & \ldots, & x_{n-1}, & x_n \\ x_2, & \ldots, & x_n, & ax_1 \end{pmatrix}$. We first discuss the centralizer F_{c_a} of c_a in the symmetry involving only the variables that c_a moves. Let D be the centralizer of a in H. Then an element y of F_{c_a} has the form

$$y = \begin{pmatrix} x_1, & \ldots, & x_{n-j+1}, & x_{n-j+2}, & \ldots, & x_n \\ kx_j, & \ldots, & kx_n, & kax_1, & \ldots, & kax_{j-1} \end{pmatrix}$$

where k is any element of D. The element y can be written as $\{k\}c_a^{j-1} = c_a^{j-1}\{k\}$. Therefore, since $c_a^n = \{a\}$ and a is in D, F_{c_a} is isomorphic to a cyclic extension of degree n of a group isomorphic to D.

If F_{y_1} denotes the centralizer of y_1 where y_1 is a substitution which is the (finite) product of cycles $c_1, \ldots, c_k$ of the same length and the same determinant class, then F_{y_1} is isomorphic to the symmetry $\Sigma(F_c;\ k,\ k+1,\ k+1)$ where F_c is the centralizer of a single cycle c.

We shall now find the centralizer F_c of a cycle c of length d. It has already been demonstrated that c may be transformed such that $c = \begin{pmatrix} \ldots, & x_{-1}, x_0, x_1, \ldots \\ \ldots, & x_0, x_1, x_2, \ldots \end{pmatrix}$.

It is clear that we need consider only the symmetry involving the variables that c moves. When c is transformed by $y = \begin{pmatrix} \ldots, & x_{-1} & ; & x_0 & ; & x_1 & ; & \ldots \\ \ldots, & h_{-1}x_{i_{-1}} & ; & h_0x_{i_0} & ; & h_1x_{i_1} & ; & \ldots \end{pmatrix}$

a computation shows that $y^{-1}cy =$

$$\begin{pmatrix} \ldots, & x_{i_{-1}}, & x_{i_0}, & x_{i_1}, & \ldots \\ \ldots, & h_{-1}^{-1}h_0x_{i_0} & , h_0^{-1}h_1x_{i_1} & , h_1^{-1}h_2x_{i_2}, & \ldots \end{pmatrix}.$$

If y is to belong to F_c the x's of this result must be the same as the ones that c moves and this gives a condition on y such that y has the form

$$y = \begin{pmatrix} \ldots, & x_{-1} & , & x_0, & x_1 & , & \ldots \\ \ldots, & k_{-1}x_{j-1}, & & k_0x_j, & k_1x_{j+1}, & & \ldots \end{pmatrix}.$$

The factors of y may now be obtained. A computation of $y^{-1}cy$ using the new form shows that

$$y^{-1}cy = \begin{pmatrix} \ldots, & x_{j-1}, & x_j & , & x_{j+1}, \ldots \\ \ldots, & k_{-1}^{-1}k_0x_j & , k_0^{-1}k_1x_{j+1}, & & k_1^{-1}k_2x_{j+2}, \ldots \end{pmatrix}.$$

In order for this result to be c we let k_0 be arbitrary in H and it follows that $k_i = k_0$ for $i = 1, -1, 2, -2, \ldots$. The final form for y to belong to F_c is then given by

$$y = \begin{pmatrix} \ldots, & x_{-1} , & x_0, & x_1 , & \ldots \\ \ldots, & kx_{j-1}, & kx_j, & kx_{j+1}, & \ldots \end{pmatrix} .$$

The powers of c always belong to F_c. A computation shows that $y = \{k\}c^j = c^j\{k\}$ where $\{k\}$ is not a true scalar but is a multiplication with k as factor in the positions corresponding to x that c moves and e as factor elsewhere. This shows that F_c is isomorphic to the direct product $H \times Z$ where Z is the infinite cyclic group, and this is independent of c (up to an isomorphism).

We shall now determine the centralizer F_y of $y = \prod_\alpha c_\alpha$ where c_α are cycles of length d in the symmetry of degree corresponding to the number of x involved. Let α run over a set of cardinal C where $1 \leq C \leq B$. Then the number of variables that appear in y is (dC). Any permutation of the c_α among themselves belong to F_y. An element y_1 of F_y will have the form $y_1 = (\prod_\alpha f_\alpha)s$ where f_α belongs to the centralizer of c_α in the symmetry on its variables. That is, f_α belongs to F_{c_α} . It is clear that all of the F_{c_α} are isomorphic since each is isomorphic to $H \times Z$. Furthermore, s is an element of the symmetric group $S(C,C^+)$. So F_y is isomorphic to the symmetry $\Sigma(F_{c_\alpha}; C, C^+, C^+)$

where F_{c_α} is the same for all α. We note that in making the computation $y_1^{-1}yy_1$ the inverse of y_1 can be found by starting on the left since the cycles of y_1 are disjoint.

Consider $y = \prod_\alpha c_\alpha$ where the c_α are finite cycles of the same length n which have the same determinant class. We shall determine F_y in the symmetry of degree corresponding to the number of x involved. Let α run over a set of cardinal C where $1 \leq C \leq B$. The number of variables involved is nC. Any permutation of the c_α is an element of F_y. An element y_1 of F_y will have the form $y_1 = (\prod_\alpha f_\alpha)s$ where f_α belongs to the centralizer of c_α in the symmetry on its variables. Furthermore, s is an element of the symmetric group $S(C,C^+)$. Let F_{c_α} be the centralizer of c_α where c_α is any of the c_α under discussion. An element y_2 of F_{c_α} is of the form $y_2 = \{k\}c_{a_\alpha}^{j-1}$ where k belongs to the centralizer D_α of a_α in H. All the c_α are of the same determinant class so the factors a_α in the $n^{\underline{th}}$ position of the c_α are conjugate. But conjugate elements have conjugate centralizers. The D_α are conjugate and, therefore, isomorphic. This shows that F_y is isomorphic to the symmetry $\Sigma(F_{c_\alpha}; C, C^+, C^+)$.

We have determined the centralizer of any element y of $\Sigma(H; B, B^+, B^+)$.

<u>Theorem 1</u>: Let y be an element of $\Sigma(H;\ B,\ B^+,\ B^+)$ and let y be written in the normal form $y = \prod_\alpha y_\alpha$, $y_\alpha = \prod_{\beta(\alpha)} c^\alpha_{\beta(\alpha)}$, where for a fixed α the $c^\alpha_{\beta(\alpha)}$ are the normalized cycles of the same length L_α (and the same determinant class a_α if $L_\alpha < d$). Let $\beta(\alpha)$ run over a set of cardinal $C_{\beta(\alpha)}$ where $0 \leq C_{\beta(\alpha)} \leq B$. Then the centralizer F_y of y in $\Sigma(H;\ B,\ B^+,\ B^+)$ is isomorphic to the strong direct product of symmetries

$$F_y \cong \prod_\alpha \Sigma(F_{c_{\beta(\alpha)}};\ C_{\beta(\alpha)},\ C^+_{\beta(\alpha)},\ C^+_{\beta(\alpha)})$$

where $F_{c_{\beta(\alpha)}}$ is the centralizer of a single cycle c^α_ε in $\Sigma(H;\ L_\beta C_{\beta(\alpha)}, (L_\beta C_{\beta(\alpha)})^+, (L_\beta C_{\beta(\alpha)})^+)$. The group $F_{c_{\beta(\alpha)}}$ consists of all elements y_1 of the form $y_1 = \{k_\alpha\}(c^\alpha_1)^j$ where k belongs to the centralizer of a_α in H (k belongs to H if $C_{\beta(\alpha)}$ is a d cycle).

For elements of the group $\Sigma(H;\ B,\ B^+,\ C)$ where $d \leq C \leq B$ the result is the same. When y is written in its cyclic decomposition, the cycles are still of length n or d and all the previous argument is valid including a revised statement of the theorem with $\Sigma(H;\ B,\ B^+,\ B^+)$ replaced by $\Sigma(H;\ B,\ B^+,\ C)$.

The elements y of the groups $\Sigma_A(H;\ B,\ B^+,\ d)$, $\Sigma(H;\ B,\ d,\ d)$, $\Sigma_A(H;\ B,\ d,\ d)$, and $\Sigma_A(H;\ n,\ n+1,\ n+1)$

have only finite cycles in their decomposition. We restate the theorem for these cases.

<u>Theorem 2</u>: Let y be an element of one of the groups $\Sigma_A(H;\ B,\ B^+,\ d)$, $\Sigma(H;\ B,\ d,\ d)$, $\Sigma_A(H;\ B,\ d,\ d)$, $\Sigma_A(H;\ n,\ n+1,\ n+1)$ and let y be written in the normal form $y = \prod_\alpha y_\alpha$, $y_\alpha = \prod_{\beta(\alpha)} c^\alpha_{\beta(\alpha)}$, where for a fixed α the $c^\alpha_{\beta(\alpha)}$ are the normalized cycles of the same length L_β and the same determinant class a_α. Let $\beta(\alpha)$ run over a set of cardinal $C_{\beta(\alpha)}$ where $0 \leq C_{\beta(\alpha)} \leq B$. The centralizer of y in its group Σ is isomorphic to the strong direct product of symmetries

$$F_y \cong \prod_\alpha \Sigma(F_{c_{\beta(\alpha)}};\ C_{\beta(\alpha)},\ C^+_{\beta(\alpha)},\ C^+_{\beta(\alpha)})$$

where $F_{c_{\beta(\alpha)}}$ is the centralizer of a single cycle c^α_ε in $\Sigma(H;\ L_\beta C_{\beta(\alpha)}, (L_\beta C_{\beta(\alpha)})^+, (L_\beta C_{\beta(\alpha)})^+)$. The group $F_{c_{\beta(\alpha)}}$ consists of all elements y_1 of the form $y_1 = \{k_\alpha\}(c^\alpha_1)^j$ where k belongs to the centralizer of a_α in H.

CHAPTER III

Splitting of the Symmetry

The investigations of this chapter are based upon the following remarks. A group G containing a normal subgroup N is said to <u>split</u> over N if there exists a subgroup M such that $G = N \cup M$, $N \cap M = E$. In this representation the group M can be replaced by any of its conjugates and the relations will still hold. For if $N \cap (yMy^{-1}) \neq E$ there exists an element g such that $g = ymy^{-1} = n$. But $m = y^{-1}ny = n_1$ by N normal, contradicting $N \cap M = E$. Furthermore, if $N \cup (yMy^{-1}) \neq G$, there exists an element g of G not contained in the union. Since $y^{-1}gy = nm$, $g = yny^{-1}ymy^{-1} = n_1ymy^{-1}$ contradicting $N \cup (yMy^{-1}) \neq G$. There may, however, exist other groups Q such that $G = N \cup Q$, $N \cap Q = E$, and such that Q is not conjugate to M. This leads to a division of all representative groups M into classes, each consisting of conjugate groups. When there is only one class, hence when all M are conjugate, we say that G <u>splits regularly</u> over N.

1. <u>The Splitting of $\Sigma(H;\ B,\ B^+,\ C)$</u>

Let H be a given fixed group, B a fixed infinite cardinal, and let C be such that $d \leq C \leq B^+$. We have already seen that $\Sigma(H;B,B^+,C) = V(B,B^+) \cup S(B,C)$, $V(B,B^+) \cap S(B,C) = E$, and hence that Σ splits over V. We shall now consider the problem of finding all groups T such that $\Sigma(H;B,B^+,C) = V(B,B^+) \cup T$, $V(B,B^+) \cap T = E$.

It is clear that if there exists such a T it is isomorphic to $S(B,C)$. Let us denote by θ the natural isomorphism between the two groups. This isomorphism is such that for each s of $S(B,C)$ $s\theta = t$ is obtained by multiplying s by an element of the basis group $V(B,B^+)$. That is, for each s of $S(B,C)$ we may write $s\theta = vs = t$.

The group $S(B,C)$ contains B elements of the form $s = (1,\alpha)$ where $\alpha = 2, 3, \ldots$. Thus T must contain B elements of the form

$$t_\alpha = (1,\alpha)\theta = \{h_{1,\alpha}, h_{2,\alpha}, \ldots, h_{\varepsilon,\alpha}, \ldots\}(1,\alpha).$$

Let us transform T by the multiplication

$$v = \{h_1, h_2, \ldots, h_\varepsilon, \ldots\}.$$

Then the group $T_\circ = vTv^{-1}$ contains the elements

$$\overset{\circ}{t}_\alpha = vt_\alpha v^{-1} =$$

$$\{h_1h_{1,\alpha}h_\alpha^{-1}, h_2h_{2,\alpha}h_2^{-1}, \ldots, h_\alpha h_{\alpha,\alpha}h_1^{-1}, \ldots, h_\varepsilon h_{\varepsilon,\alpha}h_\varepsilon^{-1}, \ldots\}(1,\alpha).$$

This shows that by a proper choice of the h_α, namely $h_\alpha = h_1h_{1,\alpha}$, it is possible to make the first factor of all the $\overset{\circ}{t}_\alpha$ the identity of H. We shall now work with the group $T_\circ$ where $T_\circ = vTv^{-1}$ and v has been chosen in the above indicated manner.

In the group $S(B,C)$ the element $(1,\alpha)$ has the property $(1,\alpha)(1,\alpha) = I$. This means that $((1,\alpha)(1,\alpha))\theta = E$; thus, $(1,\alpha)\theta(1,\alpha)\theta = \overset{\circ}{t}_\alpha\overset{\circ}{t}_\alpha = E$. But if we rename the factors of $\overset{\circ}{t}_\alpha$ so that

$$\overset{\circ}{t}_\alpha = \{e, h_{2,\alpha}, \ldots, h_{\varepsilon,\alpha}, \ldots\}(1,\alpha),$$

a computation shows that the following relations must hold

(1) $h_{\alpha,\alpha} = e$,

(2) $h^2_{\varepsilon,\alpha} = e$ for $\varepsilon \neq 1$, $\varepsilon \neq \alpha$.

Now let s be any element of S(B,C) which has the property that it moves x_1. s then has the form

$$s = \begin{pmatrix} x_1 , & \dots, & x_\alpha, & \dots, & x_\varepsilon , & \dots \\ x_{i_1} , & \dots, & x_1, & \dots, & x_{i_\varepsilon} , & \dots \end{pmatrix}$$

where the element s sends into x_1 has been denoted by x_α; $\alpha \neq 1$. We can rewrite s uniquely as follows

$$s = (1,\alpha) \begin{pmatrix} x_1, & \dots, & x_\alpha , & \dots, & x_\varepsilon , & \dots \\ x_1, & \dots, & x_{i_1} , & \dots, & x_{i_\varepsilon} , & \dots \end{pmatrix} .$$

We have already partially described the image of $(1,\alpha)$ under θ, and the above shows that to find the image of any element of S(B,C) it is sufficient to discuss those elements that leave x_1 fixed before returning to elements of the form $(1,\alpha)$.

Denote by $S_1(B,C)$ that subgroup of S(B,C) whose elements have the property that they do not move x_1. We shall now let s be an arbitrary element of $S_1(B,C)$ and completely determine $s\theta$. We shall first discuss those elements of $S_1(B,C)$ that have the property that they also leave fixed x_α for some α other than 1. This means that the elements of the present discussion are of the form

$$s_1 = \begin{pmatrix} x_1, & \dots, & x_\alpha, & \dots, & x_\varepsilon , & \dots \\ x_1, & \dots, & x_\alpha, & \dots, & x_{i_\varepsilon} , & \dots \end{pmatrix} .$$

Let us rewrite s_1 in the two following different ways

$$s_1 = (1,\alpha) \begin{pmatrix} x_1, & \dots, & x_\alpha, & \dots, & x_\varepsilon , & \dots \\ x_\alpha, & \dots, & x_1, & \dots, & x_{i_\varepsilon} , & \dots \end{pmatrix} = (1,\alpha)s,$$

$$s_1 = \begin{pmatrix} x_1, & \dots, & x_\alpha, & \dots, & x_\varepsilon , & \dots \\ x_\alpha, & \dots, & x_1, & \dots, & x_{i_\varepsilon} , & \dots \end{pmatrix} (1,\alpha) = s(1,\alpha).$$

By our previous discussion of $(1,\alpha)\theta$ we know that the factors in the first and $\alpha^{\underline{th}}$ portions are e. $(1,\alpha)\theta = \{e, h_{2,\alpha}, \ldots, e, \ldots, h_{\varepsilon,\alpha}, \ldots\}(1,\alpha)$. For every element s of S(B,C) we know that $s\theta = vs$ for some v of the basis group. Let us write $s\theta = \{k_1, k_2, \ldots, k_\varepsilon, \ldots\}s$. We know that θ is an isomorphism so we must have the two following results:

$$s_1\theta = (1,\alpha)\theta s\theta = \begin{pmatrix} x_1, \ldots, & x_\alpha, \ldots, & x_\varepsilon, \ldots \\ k_\alpha x_1, \ldots, & k_1 x_\alpha, \ldots, & h_{\varepsilon,\alpha} k_\varepsilon x_{i_\varepsilon}, \ldots \end{pmatrix},$$

$$s_1\theta = s\theta(1,\alpha)\theta = \begin{pmatrix} x_1, \ldots, & x_\alpha, \ldots, & x_\varepsilon, \ldots \\ k_1 x_1, \ldots, & k_\alpha x_\alpha, \ldots, & k_\varepsilon h_{i_\varepsilon,\alpha} x_{i_\varepsilon}, \ldots \end{pmatrix}.$$

This shows that if s_1 belongs to $S_1(B,C)$ then the factors of v, where $s_1\theta = vs_1$, in the positions corresponding to those x which s_1 leaves fixed are equal to the first factor of v.

We shall now finish discussing the present case by determining those factors of v which occur in positions corresponding to those x which s_1 moves. In order to do this we must first have the following result.

<u>Lemma</u>: Let s belong to S(B,C) and have the following properties: s moves x_1, s sends at least one x into itself, and we denote by x_β the x which s sends into x_1. Then s has the following form:

$$s = \begin{pmatrix} x_1, & \ldots, & x_\beta, & \ldots, & x_\alpha, & \ldots, & x_\varepsilon, & \ldots \\ x_\delta, & \ldots, & x_1, & \ldots, & x_\alpha, & \ldots, & x_{i_\varepsilon}, & \ldots \end{pmatrix}$$

where $\delta \neq 1$. Let $s\theta = vs$ where v is some element of $V(B,B^+)$. Then the factors which occur in the first and $\beta^{\underline{th}}$ positions of v will be equal.

Proof: It is possible to write s as described above in the following two ways:

$$s = (1,\beta)\begin{pmatrix} x_1,\ldots, & x_\beta,\ldots, & x_\alpha,\ldots, & x_\varepsilon & ,\ldots \\ x_1,\ldots, & x_\delta,\ldots, & x_\alpha,\ldots, & x_{i_\varepsilon} & ,\ldots \end{pmatrix} = (1,\beta)s_1,$$

$$s = \begin{pmatrix} x_1,\ldots, & x_\beta,\ldots, & x_\alpha,\ldots, & x_\varepsilon & ,\ldots \\ x_1,\ldots, & x_\delta,\ldots, & x_\alpha,\ldots, & x_{i_\varepsilon} & ,\ldots \end{pmatrix}(1,\delta) = s_1(1,\delta).$$

But s_1 is of the form discussed earlier. It has x_1 and x_α fixed, and we know that if $s_1\theta = v_1s_1$ then v_1 has the same factor in the first and $\alpha^{\underline{th}}$ positions. We also have information about the images of $(1,\beta)$ and $(1,\delta)$. We know

$s_1\theta = \{h_\alpha,\ldots, h_\beta,\ldots, h_\alpha,\ldots, h_\varepsilon,\ldots, h_\delta,\ldots\}s_1,$

$(1,\beta)\theta = \{e,\ldots,e,\ldots,h_{\alpha,\beta},\ldots,h_{\varepsilon,\beta},\ldots,h_{\delta,\beta},\ldots\}(1,\beta),$

$(1,\delta)\theta = \{e,\ldots,h_{\beta,\delta},\ldots,h_{\alpha,\delta},\ldots,h_{\varepsilon,\delta},\ldots,e,\ldots\}(1,\delta),$

where in $(1,\delta)\theta = v(1,\delta)$ the e occurs in the $\delta^{\underline{th}}$ position. Now using the decompositions of s above and the fact that θ is an automorphism we can say that $s\theta = (1,\beta)\theta s_1\theta = s_1\theta(1,\delta)\theta$. When these two quantities are computed the results are:

$$s\theta = \begin{pmatrix} x_1,\ldots, & x_\beta,\ldots, & x_\alpha,\ldots, & x_\varepsilon & ,\ldots \\ h_\beta x_\delta,\ldots, & h_\alpha x_1,\ldots, & h_{\alpha,\beta}h_\alpha x_\alpha,\ldots, & h_{\varepsilon,\beta}h_\varepsilon x_{i_\varepsilon} & ,\ldots \end{pmatrix},$$

$$s\theta = \begin{pmatrix} x_1,\ldots, & x_\beta,\ldots, & x_\alpha,\ldots, & x_\varepsilon & ,\ldots \\ h_\alpha x_\delta,\ldots, & h_\beta x_1,\ldots, & h_\alpha h_{\alpha,\delta}x_\alpha,\ldots, & h_\varepsilon h_{i_\varepsilon,\delta}x_{i_\varepsilon} & ,\ldots \end{pmatrix}.$$

We return to the discussion of elements of $S_1(B, C)$. We shall show that, if s_1 is an element of $S_1(B, C)$ and $s_1\theta = vs_1$, then v is a scalar. We have already determined that the factors of v that occupy positions corresponding to x which s_1 leaves fixed are equal to the first factor. We will have completed showing that v is a scalar, therefore, if we show that those factors of v which occupy positions corresponding to x which s_1 moves are also the same as the first factor of v.

Let s_1 be an element of $S_1(B,C)$ and have the following form

$$s_1 = \begin{pmatrix} x_1, & \ldots, & x_\alpha, & \ldots, & x_\beta, & \ldots \\ x_1, & \ldots, & x_\alpha, & \ldots, & x_\delta, & \ldots \end{pmatrix}$$

where $\beta \neq \delta$. Let $s_1 e = vs_1$. We already know that the factors of v in the first and $\alpha^{\underline{th}}$ positions are equal. We shall now show that the factor in the $\beta^{\underline{th}}$ position is the same as the first factor. Let us rewrite s_1 in the following fashion:

$$s_1 = \begin{pmatrix} x_1, & \ldots, & x_\alpha, & \ldots, & x_\beta, & \ldots \\ x_\delta, & \ldots, & x_\alpha, & \ldots, & x_1, & \ldots \end{pmatrix} (1,\delta) = s(1,\delta).$$

We know that $se = vs$ for some v, and by the Lemma we know that the factors of v which occupy the first and $\beta^{\underline{th}}$ positions are the same. We can, therefore, write that $se = \{k_1, \ldots, k_\alpha, \ldots, k_1, \ldots\}s$. We can also write $(1,\delta)e = \{e, h_{2,\delta}, \ldots, h_{\varepsilon,\delta}, \ldots, e, \ldots\}$ where e occurs in the first and $\delta^{\underline{th}}$ positions. Since s_1 may be decomposed as indicated above, and since e is an isomorphism, we may compute $s_1 e$ as follows:

$$s_1 e = se(1,\delta)e = \begin{pmatrix} x_1, & \ldots, & x_\alpha, & \ldots, & x_\beta, & \ldots \\ k_1 x_1, & \ldots, & k_\alpha h_{\alpha,\delta} x_\alpha, & \ldots, & k_1 x_\delta, & \ldots \end{pmatrix}.$$

This shows the factor in the first position of v where

$s_1 e = vs_1$ is the same as the factor in a position corresponding to an x which s_1 moves.

In summary of the work to this point, we can say that if s_1 is an element of $S_1(B,C)$ and if s_1 has the additional property that it leaves some x_α fixed for $\alpha \neq 1$, then $s_1 e = vs_1$ and v is a scalar.

Let us turn our attention now to those elements of $S_1(B,C)$ which do not leave fixed any x other than x_1. If s_1 belongs to $S_1(B,C)$ and has the additional property that it moves all other x, then we may write s_1 as follows

$$s_1 = \begin{pmatrix} x_1, & x_2, & \dots, & x_\varepsilon & , & \dots, & x_\beta, & \dots \\ x_1, & x_\beta, & \dots, & x_{i_\varepsilon} & , & \dots, & x_\alpha, & \dots \end{pmatrix}$$

where $\beta \neq 2$ and $\alpha \neq \beta$. Then we can rewrite s_1 as

$$s_1 = (2,\beta)\begin{pmatrix} x_1, & x_2, & \dots, & x_\varepsilon & , & \dots, & x_\beta, & \dots \\ x_1, & x_\alpha, & \dots, & x_{i_\varepsilon} & , & \dots, & x_\beta, & \dots \end{pmatrix} = \overset{\circ}{s}_1 \overset{\prime}{s}_1 .$$

But both $\overset{\circ}{s}_1$ and $\overset{\prime}{s}_1$ have the property that they belong to $S_1(B,C)$ and that they leave at least one other x fixed. Thus if $\overset{\circ}{s}_1 e = \overset{\circ}{v}_1 \overset{\circ}{s}_1$ and $\overset{\prime}{s}_1 e = \overset{\prime}{v}_1 \overset{\prime}{s}_1$ we know that both $\overset{\circ}{v}_1$ and $\overset{\prime}{v}_1$ are scalars. This means that $s_1 e = \overset{\circ}{s}_1 e \overset{\prime}{s}_1 e = \overset{\circ}{v}_1 \overset{\circ}{s}_1 \overset{\prime}{v}_1 \overset{\prime}{s}_1 = \overset{\circ}{v}_1 \overset{\prime}{v}_1 \overset{\circ}{s}_1 \overset{\prime}{s}_1 = \overset{\circ}{v}_1 \overset{\prime}{v}_1 s_1$ since scalars commute with permutations. It is also clear that $\overset{\circ}{v}_1 \overset{\prime}{v}_1$ is again a scalar; so we have that $s_1 e = vs_1$ where v is a scalar.

It is now possible to establish a homomorphism between the group $S_1(B,C)$ and a subgroup G of H in the following manner. It has been demonstrated that if s_1 is an element of $S_1(B,C)$ then $s_1 e = vs_1$ where v is a scalar, say $v = \{h_{s_1}\}$. Now let us define a correspondence ϕ between the elements of $S_1(B,C)$ and elements of H by $s_1 \phi = h_{s_1}$

where $s_1\theta = \{h_{s_1}\}s_1$. It is clear that ϕ is defined for all elements of $S_1(B,C)$ and that it is single-valued since θ was an isomorphism. ϕ preserves multiplication since scalars commute with permutations.

We have now completely determined the nature of the images of elements of $S_1(B,C)$ under the isomorphism θ. We have also established a homomorphism ϕ from $S_1(B,C)$ onto a subgroup G of H. Let us return to the problem of determining the factors of the multiplication v of the expression $(1,\alpha)\theta = v(1,\alpha)$. All we have established thus far is that the factors in the first and $\alpha^{\underline{th}}$ positions are e.

Let $(1,\alpha)\theta = \{e,\ldots, h_{\varepsilon,\alpha},\ldots, e,\ldots\}(1,\alpha)$ and let it be required to find an arbitrary factor $h_{\beta,\alpha}$ where $\beta \neq \alpha$, $\beta \neq 1$. It has been established that $(1,\beta)\theta = \{e, \ldots, h_{\varepsilon,\beta}, \ldots, e, \ldots\}(1,\beta)$ where e occurs in the first and $\beta^{\underline{th}}$ positions. It has also been established that for the element (β,α) of $S_1(B,C)$ $(\beta,\alpha)\phi = g_{\beta,\alpha}$ for some g of G. We shall establish that $h_{\beta,\alpha} = g_{\beta,\alpha}$. If one computes $(1,\alpha)(1,\beta)(1,\alpha)\theta = (1,\alpha)\theta(1,\beta)\theta(1,\alpha)\theta$, the result is

$$(1,\alpha)(1,\beta)(1,\alpha)\theta =$$

$$\begin{pmatrix} x_1,\ldots, & x_\alpha,\ldots, & x_\beta,\ldots, & x_\varepsilon,\ldots \\ h_{\alpha,\beta}x_1,\ldots, & h_{\beta,\alpha}x_\beta,\ldots, & h_{\beta,\alpha}x_\alpha,\ldots, & h_{\varepsilon,\alpha}h_{\varepsilon,\beta}h_{\varepsilon,\alpha}x_\varepsilon,\ldots \end{pmatrix}.$$

Since $(1,\alpha)(1,\beta)(1,\alpha) = (\alpha,\beta)$, and since $(\alpha,\beta)\theta = \{g_{\beta,\alpha}\}(\alpha,\beta)$, we see that $h_{\alpha,\beta} = h_{\beta,\alpha} = g_{\beta,\alpha}$. This shows that the $\beta^{\underline{th}}$ factor in the expression $(1,\alpha)\theta = v(1,\alpha)$ is simply the homomorphic image of (β,α) under ϕ.

This leads to the following theorem:

<u>Theorem 1</u>: The symmetry $\Sigma(H;B,B^+,C)$ splits over the basis group, $\Sigma(H;B,B^+,C) = V(B,B^+) \cup T$, $V(B,B^+) \cap T = E$. Any group T is the conjugate of some group $T_\circ$ obtained by the following construction. Let G be a subgroup of H that is the homomorphic image of $S_1(B,C)$ where $d \leq C \leq B^+$. Let $s\phi = g_s$ indicate the homomorphism. In particular, $(\alpha,\beta)\phi = g_{\alpha,\beta}$. Then the elements of $T_\circ$ are obtained from the elements of $S(B,C)$ by the isomorphism defined as follows:

$s* = \{g_s\}s$ for s belonging to $S_1(B,C)$,

$(1,\alpha)* = \{e, g_{2,\alpha},\ldots,g_{\varepsilon,\alpha},\ldots,e,\ldots\}(1,\alpha)$

where e occurs in the first and α<u>th</u> positions.

It has already been shown that any group T, after suitable transformation must have the form indicated in the correspondence above. It remains to show that the set of substitutions defined by the correspondence above actually form a group isomorphic to $S(B,C)$.

Any element s of $S(B,C)$ may be written uniquely in the form $s = (1,\alpha)s_1$ where s_1 is an element of $S_1(B,C)$. For if s belongs to $S(B,C)$ and s belongs also to $S_1(B,C)$, we can write $s = (1,1)s$. If s belongs to $S(B,C)$ and s does not belong to $S_1(B,C)$, then $s = \begin{pmatrix} x_1, & \ldots, & x_\alpha, & \ldots \\ x_\beta, & \ldots, & x_1, & \ldots \end{pmatrix} = (1,\alpha) \begin{pmatrix} x_1, & \ldots, & x_\alpha, & \ldots \\ x_1, & \ldots, & x_\beta, & \ldots \end{pmatrix}$.

The correspondence of the theorem defines a unique substitution s^* corresponding to each s of $S(B,C)$ by $s^* = (1,\alpha)^* s_1{}^*$ or

$$s^* = \{e, g_{2,\alpha}, \ldots, g_{\varepsilon,\alpha}, \ldots, e, \ldots\}(1,\alpha)\{g_{s_1}\}s_1$$

where e occurs in the first and $\alpha^{\underline{th}}$ positions, and, therefore,

$$s^* = \{g_{s_1},\ g_{2,\alpha}g_{s_1}, \ldots,\ g_{\varepsilon,\alpha}g_{s_1}, \ldots,\ g_{s_1}, \ldots\}s.$$

We see that the correspondence is one-to-one. For if s and $s_\circ$ are different elements of $S(B,C)$, then s^* and $s_\circ{}^*$ will differ in their permutation. If two distinct images t, $t_\circ$ are considered, and if they differ in the permutation part, then the pre-images are different. It will not happen that $t = vs$ and $t_\circ = v_\circ s$ where the permutations are the same and the multiplications are different, because the factors of t and $t_\circ$ are determined by the unique decomposition of s into $(1,\alpha)s_1$.

We shall now prove that the so defined correspondence preserves multiplication. Let $\bar{s} = (1,\beta)\bar{s}_1$ be another element of $S(B,C)$ that has been written in the normal form with $\bar{s}_1$ belonging to $S_1(B,C)$. We shall then prove that

(1) $(s\bar{s})^* = s^*\bar{s}^*$.

First we notice that it is no limitation to discuss only the case where $\bar{s}$ is simply $(1,\beta)$, because by use of the definition above we can show that $(s\bar{s}_1)^* = s^*\bar{s}_1{}^*$ for arbitrary s of $S(B,C)$ and any $\bar{s}_1$ of $S_1(B,C)$. To verify that $(s\bar{s}_1)^* = s^*\bar{s}_1{}^*$ let us write $s = (1,\alpha)s_1$ and compute both $(s\bar{s}_1)^*$ and $s^*\bar{s}_1{}^*$.

$(s\bar{s}_1)^* = ((1,\alpha)s_1\bar{s}_1)^* = ((1,\alpha)\bar{\bar{s}}_1)^* = (1,\alpha)^*\bar{\bar{s}}_1{}^*$

$s^*\bar{s}_1{}^* = ((1,\alpha)s_1)^*\bar{s}_1{}^* = (1,\alpha)^*s_1{}^*\bar{s}_1{}^*$

But $\bar{\bar{s}}_1{}^* = s_1{}^*\bar{s}_1{}^*$, for we know that $s_1{}^* = \{g_{s_1}\}s_1$ and $\bar{s}_1{}^* = \{g_{\bar{s}_1}\}\bar{s}_1$. Then $s_1{}^*\bar{s}_1{}^* = \{g_{s_1}\}s_1\{g_{\bar{s}_1}\}\bar{s}_1 = \{g_{s_1}g_{\bar{s}_1}\}\bar{\bar{s}}_1$. It follows from the fact that Φ is a homomorphism that $g_{s_1}g_{\bar{s}_1} = g_{s_1\bar{s}_1} = g_{\bar{\bar{s}}_1}$.

We wish to apply the results of the above paragraph to (1) to show that it is no limitation to choose $\bar{s}$ as $(1,\beta)$. We know that $(s\bar{s})^* = (s(1,\beta)\bar{s}_1)^* = (s(1,\beta))^*\bar{s}_1{}^*$ by the above work. If we knew $(s(1,\beta))^* = s^*(1,\beta)^*$, we would have $(s\bar{s})^* = s^*(1,\beta)^*\bar{s}_1{}^* = s^*\bar{s}^*$. We shall have to prove, therefore, that

(2) $(s(1,\beta))^* = s^*(1,\beta)^* = (1,\alpha)^*s_1{}^*(1,\beta)^*$.

In order to establish (2) we note that it will follow if we can prove the two simpler relations

(2.1) $((1,\alpha)s)^* = (1,\alpha)^*s^*$ for any s belonging to $S(B,C)$

(2.2) $(s_1s)^* = s_1{}^*s^*$ for any s_1 of $S_1(B,C)$ and any s of $S(B,C)$.

For if we had (2.1) and (2.2) we could write $(s(1,\beta))^* = ((1,\alpha)s_1(1,\beta))^* = (1,\alpha)^*(s_1(1,\beta))^* = (1,\alpha)^*s_1{}^*(1,\beta)^* = ((1,\alpha)s_1)^*(1,\beta)^* = s^*(1,\beta)^*$.

Let us prove (2.1) first. By our earlier discussion we know it is no limitation to take s as $(1,\beta)$. Therefore, we shall establish

(2.1)' $((1,\alpha)(1,\beta))^* = (1,\alpha)^*(1,\beta)^*$.

Case 1. $\alpha = \beta$

From the definition of the correspondence we have $(1,\alpha)* = \{e, g_{2,\alpha}, \ldots, g_{\varepsilon,\alpha}, \ldots, e, \ldots\}(1,\alpha)$. A computation shows that

$$(1,\alpha)*(1,\alpha)* = \{e, g^2_{2,\alpha}, \ldots, g^2_{\varepsilon,\alpha}, \ldots, e, \ldots\}.$$

But $(\varepsilon,\alpha)\phi = g_{\varepsilon,\alpha}$ and ϕ is a homomorphism. So $(\varepsilon,\alpha)^2\phi = I\phi = e = g^2_{\varepsilon,\alpha}$ for all $\varepsilon \neq 1$, $\varepsilon \neq \alpha$. This shows that $(1,\alpha)*(1,\alpha)* = E$. On the other hand, we have $((1,\alpha)(1,\alpha))* = I* = \{g_I\}I = \{e\}I = E$.

Case 2. $\alpha \neq \beta$

We have $(1,\alpha)* = \{e, \ldots, g_{\varepsilon,\alpha}, \ldots, e, \ldots\}(1,\alpha)$ where the e occurs in the first and $\alpha^{\underline{th}}$ positions. We also have $(1,\beta)* = \{e, \ldots, g_{\varepsilon,\beta}, \ldots, e, \ldots\}(1,\beta)$ with the e occuring in the first and $\beta^{\underline{th}}$ positions. When one computes $(1,\alpha)*(1,\beta)*$ the result is

$$(1,\alpha)*(1,\beta)* = \{g_{\alpha,\beta},\ldots, g_{\varepsilon,\alpha}g_{\varepsilon,\beta},\ldots, e, \ldots, g_{\beta,\alpha},\ldots\}(1,\alpha)(1,\beta)$$

where e occurs in the $\alpha^{\underline{th}}$ position and $\varepsilon \neq 1, \alpha, \beta$.

Now let us compute $((1,\alpha)(1,\beta))*$. We note first that $(1,\alpha)(1,\beta) = (1,\beta)(\alpha,\beta)$; so $((1,\alpha)(1,\beta))* = ((1,\beta)(\alpha,\beta))*$. We can apply the definition of the correspondence and compute, and we have

$$((1,\beta)(\alpha,\beta))*$$
$$= \{e, \ldots, g_{\varepsilon,\beta}, \ldots, e, \ldots\}(1,\beta)\{g_{\alpha,\beta}\}(\alpha,\beta)$$
$$= \{g_{\alpha,\beta},\ldots,g_{\varepsilon,\beta}g_{\alpha,\beta},\ldots,g^2_{\alpha,\beta},\ldots,g_{\alpha,\beta},\ldots\}(1,\alpha)(1,\beta).$$

By comparison of the two computations one sees that the factors in the $1^{\underline{st}}$, $\alpha^{\underline{th}}$, and $\beta^{\underline{th}}$ positions are the same when one uses the aforementioned result that $g_{\alpha,\beta}^2 = e$. It remains to show that $g_{\varepsilon,\alpha}g_{\varepsilon,\beta} = g_{\varepsilon,\beta}g_{\alpha,\beta}$ where ε is neither α nor β. But we observe that $(\varepsilon,\alpha)(\varepsilon,\beta) = (\varepsilon,\beta)(\alpha,\beta)$ and use once again the fact that ϕ is a homomorphism and

$$((\varepsilon,\alpha)(\varepsilon,\beta))\phi = (\varepsilon,\alpha)\phi(\varepsilon,\beta)\phi = g_{\varepsilon,\alpha}g_{\varepsilon,\beta},$$
$$((\varepsilon,\beta)(\alpha,\beta))\phi = (\varepsilon,\beta)\phi(\alpha,\beta)\phi = g_{\varepsilon,\beta}g_{\alpha,\beta}$$

must be the same. This concludes the proof of $(2.1)'$.

We shall now prove (2.2): $(s_1 s)* = s_1*s*$ for any s_1 of $S_1(B,C)$ and any s of $S(B,C)$. We saw earlier that it is no limitation to choose s as $(1,\beta)$, and we shall do so. Therefore, we shall establish:

$(2.2)'$ $(s_1(1,\beta))* = s_1*(1,\beta)*$ for any s_1 of $S_1(B,C)$.

Case 1. s_1 does not move x_β.

We have that $s_1* = \{g_{s_1}\}s_1$ and that $(1,\beta)* = v(1,\beta) = \{e, \ldots, g_{\varepsilon,\beta}, \ldots, e, \ldots\}(1,\beta)$, where e occurs in the first and $\beta^{\underline{th}}$ positions. We then compute $s_1*(1,\beta)*$ and simplify by noting that $(1,\beta)$ commutes with v, that scalars commute with all permutations, and that in the present case we are discussing $(1,\beta)$ commutes with s_1. The result of this computation is

$$s_1*(1,\beta)* =$$

$$\begin{pmatrix} x_1, & \ldots, & x_\beta, & \ldots, & x_\alpha, & \ldots, & x_\varepsilon, & \ldots \\ g_{s_1}x_\beta, & \ldots, & g_{s_1}x_1, & \ldots, & g_{s_1}g_{\alpha,\beta}x_\alpha, & \ldots, & g_{s_1}g_{\delta,\beta}x_\delta, & \ldots \end{pmatrix}$$

where instead of listing one general element we have listed two. It is necessary to look at the factors occurring in the positions occupied by the

x_α which are not moved by s_1 and also the x_ε which are moved by s_1.

We shall now compute the left hand side of $(2.2)'$. Since it cannot be computed as it is written above, we note that $(1,\beta)$ commutes with s_1 and use (2.1) to write $(s_1(1,\beta))* = ((1,\beta)s_1)* = (1,\beta)*s_1*$. We can compute $(1,\beta)*s_1*$ and simplify by using the fact that scalars commute with permutations. The result of this computation and simplification is

$(s_1(1,\beta))* =$

$$\begin{pmatrix} x_1, & \ldots, & x_\beta, & \ldots, & x_\alpha, & \ldots, & x_\varepsilon, & \ldots \\ g_{s_1}x_\beta, & \ldots, & g_{s_1}x_1, & \ldots, & g_{\alpha,\beta}g_{s_1}x_\alpha, & \ldots, & g_{\varepsilon,\beta}g_{s_1}x_\delta, & \ldots \end{pmatrix}.$$

By comparison one sees that it remains to show that $g_{\alpha,\beta}g_{s_1} = g_{s_1}g_{\alpha,\beta}$ when s_1 does not move x_α, and that $g_{\varepsilon,\beta}g_{s_1} = g_{s_1}g_{\delta,\beta}$ when s_1 sends x_ε into x_δ. We are at present discussing the case where s_1 leaves x_β fixed. If s_1 also leaves x_α fixed, then (α,β) commutes with s_1. This means, using the fact that ϕ is a homomorphism, that $g_{\alpha,\beta}g_{s_1} = ((\alpha,\beta)s_1)\phi = (s_1(\alpha,\beta))\phi = g_{s_1}g_{\alpha,\beta}$. The same type of argument is applicable to showing that $g_{\varepsilon,\beta}g_{s_1} = g_{s_1}g_{\delta,\beta}$ although not so simply stated. Let us write $s_1(\delta,\beta)$ in these two ways

$$s_1(\delta,\beta) = \begin{pmatrix} x_1, & \ldots, & x_\beta, & \ldots, & x_\varepsilon, & \ldots \\ x_1, & \ldots, & x_\beta, & \ldots, & x_\delta, & \ldots \end{pmatrix} (\delta,\beta),$$

and

$$s_1(\delta,\beta) = (\beta,\varepsilon) \begin{pmatrix} x_1, & \ldots, & x_\beta, & \ldots, & x_\varepsilon, & \ldots \\ x_1, & \ldots, & x_\beta, & \ldots, & x_\delta, & \ldots \end{pmatrix} .$$

This shows that we must have $(s_1(\delta,\beta))\phi = ((\beta,\varepsilon)s_1)\phi$.

But this means $g_{s_1} g_{\delta,\beta} = g_{\beta,\varepsilon} g_{s_1}$ which was to be shown.

Case 2. s_1 does move x_β.

Let us denote by x_α the image of x_β under s_1 and since x_β is not sent into itself by s_1, s_1 must send some other x into x_β. Let us denote this by x_δ. We know that $s_1{*} = \{g_{s_1}\}s_1$ and that $(1,\beta){*} = \{e,\ldots,g_{\varepsilon,\beta},\ldots,e,\ldots\}(1,\beta)$. We now compute the right hand side of $(2.2)'$ and find

$s_1{*}(1,\beta){*} =$

$$\begin{pmatrix} x_1, & \ldots, & x_\beta, & \ldots, & x_\delta, & \ldots, & x_\varepsilon\ , & \ldots \\ g_{s_1}x_\beta, & \ldots, & g_{s_1}g_{\alpha,\beta}x_\alpha, & \ldots, & g_{s_1}x_1, & \ldots, & g_{s_1}g_{i_\varepsilon,\beta}x_{i_\varepsilon}, & \ldots \end{pmatrix}.$$

Since it is not possible to compute the left hand side of $(2.2)'$ as it is at present, we rewrite $s_1(1,\beta)$ as follows:

$s_1(1,\beta) =$

$$\begin{pmatrix} x_1, & \ldots, & x_\beta, & \ldots, & x_\delta, & \ldots, & x_\varepsilon\ , & \ldots \\ x_1, & \ldots, & x_\alpha, & \ldots, & x_\beta, & \ldots, & x_{i_\varepsilon}, & \ldots \end{pmatrix}(1,\beta) = (1,\delta)s_1$$

and use (2.1) to compute. $(s_1(1,\beta)){*} = ((1,\delta)s_1){*} = (1,\delta){*}s_1{*}$. We know that $(1,\delta){*} = \{e,\ldots,g_{\varepsilon,\delta},\ldots,e,\ldots\}(1,\delta)$ where e occurs in the first and $\delta^{\underline{th}}$ positions. We now compute and simplify. The result is

$(s_1(1,\beta)){*} =$

$$\begin{pmatrix} x_1, & \ldots, & x_\beta, & \ldots, & x_\delta, & \ldots, & x_\varepsilon\ , & \ldots \\ g_{s_1}x_\beta, & \ldots, & g_{\beta,\delta}g_{s_1}x_\alpha, & \ldots, & g_{s_1}x_1, & \ldots, & g_{\varepsilon,\delta}g_{s_1}x_{i_\varepsilon}, & \ldots \end{pmatrix}.$$

By comparison one sees that the factors are the same in the first and $\delta^{\underline{th}}$ positions. It remains to show that $g_{\beta,\delta}g_{s_1} = g_{s_1}g_{\alpha,\beta}$ and $g_{\varepsilon,\delta}g_{s_1} = g_{s_1}g_{i_\varepsilon,\beta}$. The method of proof is the one used previously in similar cases. A computation shows that $(\beta,\delta)s_1 = s_1(\alpha,\beta)$ and since ϕ is a homomorphism we must have $g_{\beta,\delta}g_{s_1} = g_{s_1}g_{\alpha,\beta}$. A computation also shows that $(\varepsilon,\delta)s_1 = s_1(i_\varepsilon,\beta)$ and hence $g_{\varepsilon,\delta}g_{s_1} = g_{s_1}g_{i_\varepsilon,\beta}$.

This concludes showing that the correspondence defined in the theorem preserves multiplication. The images of the elements of S(B,C) form a group isomorphic to S(B,C) which we shall now call T. It is also clear that $V(B,B^+) \cap T = E$, because if there exists a common element v = t the permutation portion of t must be I. But $I\phi = e$; so t = E. We can also see that $V(B,B^+) \cup T = \Sigma(H;B,B^+,C)$. Let m be any element of Σ and m = vs where v is an element of V and s is an element of S. We shall show that $m = v_1t$ for v_1 in V and t in T. We know $s* = t = v_2s$ for some v_2 of V. Then we can write $m = vv_2^{-1}v_2s = (vv_2^{-1})v_2s = v_1t$. This concludes the proof of the theorem.

The construction of the group T_0 of the last theorem leads to a condition for Σ to split regularly.

<u>Theorem 2</u>: A necessary and sufficient condition for the symmetry $\Sigma(H;B,B^+,C)$, where $d \leq C \leq B^+$, to split regularly over its basis group is that H contain no subgroup different from e homomorphic to the symmetric group $S_1(B,C)$.

<u>Proof</u>: If H contains no subgroup different from e that is the homomorphic image of S_1, then all the factors g_s in the previous theorem are e, and T_0 is simply S.

If Σ splits regularly the group T_0 can be transformed into S. There exists an m such that $mT_0m^{-1} = S$. But this m may be assumed to be a multiplication.

For if $m = vs$ we may rewrite $m = sv_1$ since V is normal in Σ. Then $mT_\circ m^{-1} = sv_1 T_\circ v_1^{-1} s^{-1} = S$. This means that $v_1 T_\circ v_1^{-1} = s^{-1} Ss = S$. Therefore, m need have been only a multiplication, $m = \{k_1, k_2, \ldots, k_\varepsilon, \ldots\}$.

Now consider those elements t of $T_\circ$ of the form $t = \{g_s\}s$. We note that such an element has the property that it leaves x_1 fixed. When mtm^{-1} is computed, it follows that $k_1 g_s k_1^{-1} = e$, and hence $g_s = e$. This proves the group G is e.

<u>Corollary 1</u>: A necessary and sufficient condition for the symmetry $\Sigma(H;B,B^+,C)$, where $d^+ \leq C \leq B^+$, to split regularly over its basis group is that N contain no subgroup isomorphic to $S(B,C)$.

<u>Proof</u>: If Σ splits regularly then H contains no subgroup, except e, which is the homomorphic image of $S_1(B,C)$. Since $S_1(B,C)$ is isomorphic to $S(B,C)$, it follows that H contains no subgroup isomorphic to $S(B,C)$.

Let us assume that H contains no subgroup isomorphic to $S(B,C)$ and that Σ does not split regularly. Then H contains a homomorphic image G of $S_1(B,C)$. Then by Theorem 11, Chapter I, G contains a subgroup isomorphic to $S(B,C)$. A contradiction has been reached, so Σ must split regularly.

Corollary 2: A necessary and sufficient condition for $\Sigma(H;\ B,\ B^{+},\ d)$ to split regularly over its basis group is that H contain no element of order 2.

Proof: If Σ splits regularly, then by Theorem 2, H contains no subgroup, except e, which is the homomorphic image of $S_1(B,d)$. Hence, H contains no element of order 2 for such an element would generate a subgroup which would be a homomorphic image of $S_1(B,d)$.

Conversely, if H contains no element of order 2, then H contains no subgroup, except e, which is the homomorphic image of $S_1(B,d)$. For such a subgroup must be a cyclic group of order 2 or be isomorphic to $S_1(B,d)$. Therefore, by Theorem 2, Σ splits regularly.

Corollary 3: For every group H there exists a group $\Sigma(H;\ B,\ B^{+},\ B^{+})$ such that the monomial group splits regularly over the basis group.

Proof: This follows from Corollary 1 if the cardinal B is chosen such that $o(S(B,B^{+})) > o(H)$.

2. The Splitting of $\Sigma_A(H;\ n,\ n+1,\ n+1)$

We shall now discuss the splitting of the group $\Sigma_A(H;\ n,\ n+1,\ n+1)$. It was shown earlier that Σ_A splits over the basis group, $\Sigma_A(H;\ n,\ n+1,\ n+1) = V(n,\ n+1) \cup A(n,\ n+1)$, $V(n,\ n+1) \cap A(n,\ n+1) = E$. We shall consider the problem of finding all groups T such that $\Sigma_A = V \cup T$, $V \cap T = E$.

If there exists such a group T, we see that T is isomorphic to $A(n, n+1)$. Furthermore, this isomorphism, which we denote by θ, can be taken in such a way that $s\theta = t = vs$ for all s of $A(n, n+1)$ and for some v of V.

Carmichael [3, p. 31] shows that the elements $s_i = (1, i, 2)$, $i = 3, \ldots, n$, generate the group $A(n, n+1)$. T must contain elements t_i, $i = 3, \ldots, n$, such that $s_i\theta = t_i$. We shall denote these t_i as follows:

$$t_i = \{h_{1,i}h_{2,i}, \ldots, h_{j,i}, \ldots, h_{n,i}\}(1,i,2).$$

Now let us transform T by the multiplication $v = \{k_1, k_2, \ldots, k_j, \ldots, k_n\}$. Then the group $T_\circ = vTv^{-1}$ must contain the n-2 elements

$$t_i^\circ = vt_iv^{-1} =$$

$$\{k_1h_{1,i}k_i^{-1}, k_2h_{2,i}k_1^{-1}, \ldots, k_ih_{i,i}k_2^{-1}, \ldots, k_jh_{j,i}k_j^{-1}, \ldots, k_nh_{n,i}k_n^{-1}\}(1,i,2)$$

for $i = 3, \ldots, n$. We shall now choose the factors of v in such a way that the factors of t_i° are simplified. Let k_1 be an arbitrary fixed element of H and choose $k_i = k_1h_{1,i}$ for $i = 3, \ldots, n$. We are also at liberty to choose k_2, and we take $k_2 = k_1h_{2,3}^{-1}$. This shows that, for a proper choice of v, $T_\circ$ must contain the elements

$$t_3^\circ = \{e, e, g_{3,3}, g_{4,3}, \ldots, g_{j,3}, \ldots, g_{n,3}\}(1,3,2),$$

$$t_i^\circ = \{e, g_{2,i}, g_{3,i}, \ldots, g_{j,i}, \ldots, g_{n,i}\}(1,i,2)$$

for $i = 4, \ldots, n$.

We shall now work with the group $T_\circ$ instead of T. We shall express the factors of $\overset{\circ}{t}_i$ for $i = 4, \ldots, n$ in terms of the factors of $\overset{\circ}{t}_3$ by using properties of the elements s_i of A(n, n+1) and the isomorphism θ. We observe first that in the group A(n, n+1) the elements s_i have the property that $(s_i)^3 = I$. This means that $(s_i)^3\theta = (t_i)^3 = I\theta = E$. The quantities $(\overset{\circ}{t}_3)^3$ and $(\overset{\circ}{t}_i)^3$ for $i = 4, \ldots, n$ are computed and the results are:

$$(\overset{\circ}{t}_3)^3 = \{g_{3,3}, g_{3,3}, g_{3,3}, g^3_{4,3}, \ldots, g^3_{n,3}\},$$

$$(\overset{\circ}{t}_i)^3 = \{g_{i,i}g_{2,i}, g_{2,i}g_{i,i}, g^3_{3,i}, \ldots, g_{i,i}g_{2,i}, \ldots, g^3_{n,i}\}$$

where $g_{i,i}g_{2,i}$ appears in the first and $i^{\underline{th}}$ positions of $(\overset{\circ}{t}_i)^3$. The above shows that the following relations must hold:

(1) $g_{3,3} = e$,

(2) $g^3_{j,3} = e$ for $j = 4, \ldots, n$,

(3) $g_{i,i}g_{2,i} = g_{2,i}g_{i,i} = e$ for $i = 4, \ldots, n$.

We can rewrite $\overset{\circ}{t}_3$ using (1):

$$\overset{\circ}{t}_3 = \{e, e, e, g_{4,3}, \ldots, g_{n,3}\}(1,3,2).$$

Once again we examine the elements of A(n, n+1). In this group $(s_3s_i)^2 = ((1,3,2)(1,i,2))^2 = I$ for $i = 4, \ldots, n$. So in the group $T_\circ$ we must have $(\overset{\circ}{t}_3\overset{\circ}{t}_i)^2 = E$. When this is computed we see that

$(\overset{\circ}{t}_3\overset{\circ}{t}_i)^2 =$

$\{g_{3,i}g_{2,i}, g_{i,3}g_{i,i}, \ldots, g_{i,3}g_{i,i}, \ldots, (g_{n,3}g_{n,i})^2\}.$

This shows that the following conditions must hold:

(4) $g_{3,i}g_{2,i} = e$ for $i = 4, \ldots, n,$

(5) $g_{i,3}g_{i,i} = e$ for $i = 4, \ldots, n,$

(6) $(g_{j,3}g_{j,i})^2 = e$ for $j = 4, \ldots, i-1, i+1, \ldots, n$
$i = 4, \ldots, n.$

We can now combine some of these relations to give further information about the factors of $\overset{\circ}{t}_i$ for $i > 3$. From (5) and (2) we have

(7) $g_{i,3}g_{i,i} = e$, or $g_{i,i} = g_{i,3}^2$ for $i = 4, \ldots, n.$

From (7), (3), and (2) we have

(8) $g_{i,i}g_{2,i} = e$, or $g_{2,i} = g_{i,3}$ for $i = 4, \ldots, n.$

From (4), (2), and (8) we have

(9) $g_{3,i}g_{2,i} = e$, or $g_{3,i} = g_{i,3}^2$ for $i = 4, \ldots, n.$

This set of results allows us to rewrite the elements $\overset{\circ}{t}_i$ of $T_\circ$ in the following fashion:

$\overset{\circ}{t}_3 = \{e, e, e, g_{4,3}, \ldots, g_{n,3}\}(1,3,2)$

$\overset{\circ}{t}_i = \{e, g_{i,3}, g_{i,3}^2, g_{4,i}, \ldots,$

$g_{i-1,i}, g_{i,3}^2, g_{i+1,i}, \ldots, g_{n,i}\}(1,i,2).$

In order to write the other factors of $\overset{\circ}{t}_i$, $i = 4, \ldots, n$, in terms of the factors of $\overset{\circ}{t}_3$ we will use another property of the elements of $A(n, n+1)$, namely $(s_i s_j)^2 = I$ for all i, j such that $i \neq j$. It follows that $(\overset{\circ}{t}_i\overset{\circ}{t}_j)^2 = E$ for all i, j such that $i \neq j$. When $(\overset{\circ}{t}_i\overset{\circ}{t}_j)^2$ is computed the result is $(\overset{\circ}{t}_i\overset{\circ}{t}_j)^2 = \{g_{i,j}g_{i,3}^2g_{j,3}, \ldots\}$. The following relation must be satisfied:

(10) $g_{i,j}g_{i,3}^2g_{j,3} = e, \ i \neq j.$

From (10) and (2) we have

(11) $g_{i,j} = g_{j,3}^2g_{i,3}, \ i \neq j.$

When this result is applied to the elements $\overset{\circ}{t}_i$, they have the following form:

$$\overset{\circ}{t}_3 = \{e, e, e, g_{4,3}, \ldots, g_{n,3}\}(1,3,2)$$

$$\overset{\circ}{t}_i = \{e, g_{i,3}, g_{i,3}^2, \ldots, g_{i,3}^2g_{i-1,3}, g_{i,3}^2, g_{i,3}^2g_{i+1,3}, \ldots, g_{i,3}^2g_{n,3}\}(1,i,2)$$

for $i = 4, \ldots, n$.

Consider the set of elements $g_{j,3}$ of H where $j = 4, \ldots, n$. We have seen in (2) that $(g_{j,3})^3 = e$ for all j. From (6) we have that $(g_{j,3}g_{j,i})^2 = e$ for $j = 4, \ldots, i-1, i+1, \ldots, n$ and $i = 4, \ldots, n$. We will use this to show that $(g_{i,3}g_{j,3})^2 = e$ for all i, j such that $i \neq j$. In the expression $(g_{j,3}g_{j,i})^2 = e$ fix i as one of the subscripts of the index set $4, \ldots, n$. Let j have any one of the values $4, \ldots, i-1, i+1, \ldots, n$ and $j \neq i$. Then $(g_{j,3}g_{j,i})^2 = (g_{j,3}g_{i,3}^2g_{j,3})^2 = g_{j,3}g_{i,3}^2g_{j,3}^2g_{i,3}^2g_{j,3} = e$ where we have used the earlier result that $g_{j,i} = g_{i,3}^2g_{j,3}$.

Then

$$g_{i,3}^2 g_{j,3}^2 g_{i,3}^2 = g_{j,3}^4 = g_{j,3}$$

$$g_{j,3}^2 = g_{i,3} g_{j,3} g_{i,3}$$

$$e = g_{i,3} g_{j,3} g_{i,3} g_{j,3}$$

which was to be shown. This proves $(g_{i,3} g_{j,3})^2 = e$ for all i, j such that $i \neq j$ and $j = 4, \ldots, n$; $i = 4, \ldots, j-1, j+1, \ldots, n$.

In summary we may say that the elements $g_{4,3}, \ldots, g_{n,3}$ satisfy the conditions

(α) $g_{j,3}^3 = e$ for $j = 4, \ldots, n$,

(β) $(g_{i,3} g_{j,3})^2 = e$ for $i \neq j$.

It follows from Carmichael [3, pp. 166, 172] that the group generated by $g_{4,3}, \ldots, g_{n,3}$ is a homomorphic image of $A(n-1, n)$.

We are led to the following theorem:

<u>Theorem 1</u>: The group $\Sigma_A(H; n, n+1, n+1)$ splits over the basis group, $\Sigma_A(H; n, n+1, n+1) = V(n, n+1) \cup T$, $V(n, n+1) \cap T = E$. The group T is conjugate to some group T_0 obtained as follows. Let G be a subgroup of H which is the homomorphic image of $A(n-1, n)$. Let $g_4, \ldots, g_n$ be generators of G, satisfying the following relations:

(α) $g_i^3 = e$, $i = 4, \ldots, n$,

(β) $(g_i g_j)^2 = e$ where $i \neq j$.

Let $s_i = (1,i,2)$ for $i = 3, \ldots, n$ generate the group $A(n, n+1)$. Then the elements of T_0 are obtained from the elements of $A(n, n+1)$ by the isomorphism θ defined by

$$s_3\theta = \overset{\circ}{t}_3 = \{e,e,e,g_4,\ldots,g_n\}(1,3,2)$$

$$s_i\theta = \overset{\circ}{t}_i = \{e,g_i,g_i^2,g_i^2g_4,\ldots,$$
$$g_i^2g_{i-1},g_i^2,g_i^2g_{i+1},\ldots,g_i^2g_n\}(1,i,2)$$

for $i = 4, \ldots, n$.

It has already been shown that any group T after proper transformation must have the form indicated by the theorem. We shall now show that $(\overset{\circ}{t}_i)^3 = E$ for $i = 3, \ldots, n$ and $(\overset{\circ}{t}_i\overset{\circ}{t}_j)^2 = E$ for $i \neq j$.

We shall verify first that $(\overset{\circ}{t}_i)^3 = E$ for $i = 3, \ldots, n$. $(\overset{\circ}{t}_3)^3 = \{e, e, e, g_4^3, \ldots, g_n^3\} = E$ because $g_i^3 = e$ by (α). $(\overset{\circ}{t}_i)^3 = \{g_i^2g_i, g_ig_i^2, (g_i^2)^3, \ldots, (g_i^2g_{i-1})^3, g_i^2g_i, (g_i^2g_{i+1})^3, \ldots, (g_i^2g_n)^3\}$ for $i = 4, \ldots, n$. It is clear that the factors in the first, second, third, and i^{th} positions are e by (α). It remains to show $(g_i^2g_j)^3 = e$ for $j = 4, \ldots, i-1, i+1, \ldots, n$. But $(g_i^2g_j)^3 = g_i^2g_jg_i^2g_jg_i^2g_j =$
$g_ig_ig_jg_i(g_jg_j^2)g_ig_jg_ig_ig_j = g_i(g_ig_jg_ig_j)g_j(g_jg_ig_jg_i)g_ig_j$
$= e\ e\ e = e$ where we have used (β) to simplify.

We shall now show that $(\overset{\circ}{t}_i\overset{\circ}{t}_j)^2 = E$ for $i \neq j$.

Case 1. $i = 3$, $j = 4, \ldots, n$.

A simple computation shows that

$$(\overset{\circ}{t}_3\overset{\circ}{t}_j)^2 = \{g_j^2 g_j,\ g_j g_j^2,\ g_j g_j^2,\ (g_4 g_j^2 g_4)^2,\ \ldots,$$
$$(g_{j-1}g_j^2 g_{j-1})^2,\ g_j g_j^2,\ (g_{j+1}g_j^2 g_{j+1})^2,$$
$$\ldots,\ (g_n g_j^2 g_n)^2\}.$$

The factors occupying the first, second, third, and $j^{\underline{th}}$ positions are e by (α). It remains to show that $(g_k g_j^2 g_k)^2 = e$ for $k = 4, \ldots, j-1, j+1, \ldots, n$. The relations ($\alpha$) and ($\beta$) of the theorem may be combined to give the following result: $g_i g_j = g_j^2 g_i^2$. We use this and (α) to write $(g_k g_j^2 g_k)^2 = g_k g_j^2 g_k^2 g_j^2 g_k = g_k g_k g_j g_j^2 g_k = e$.

Case 2. $i \neq 3$, $j = 3$.

$(\overset{\circ}{t}_i\overset{\circ}{t}_3)^2 =$

$\{g_i g_i^2,\ g_i g_i^2,\ g_i^2 g_i, \ldots,\ (g_i^2 g_k g_k)^2, \ldots,\ g_i^2 g_i, \ldots\}$.

The factors in the first, second, third, and $i^{\underline{th}}$ positions are e. It remains to show that $(g_i^2 g_k g_k)^2 = e$ for k different from 1,2,3,i. $g_i^2 g_k^2 g_i^2 g_k^2 = g_k g_i g_k g_i = e$.

Case 3. $i \neq 3$, $j \neq 3$, $i \neq j$.

For convenience we shall assume that $i < j$. We compute $(\overset{\circ}{t}_i\overset{\circ}{t}_j)^2$ and the result is

$$(\overset{\circ}{t}_i\overset{\circ}{t}_j)^2 = \{g_j^2 g_i g_i^2 g_j, g_i g_i^2 g_j g_j^2, (g_i^2 g_j^2)^2, (g_i^2 g_4 g_j^2 g_4)^2,$$
$$\ldots, (g_i^2 g_{i-1} g_j^2 g_{i-1})^2, g_i^2 g_j g_j^2 g_i, (g_i^2 g_{i+1} g_j^2 g_{i+1})^2, \ldots,$$
$$(g_i^2 g_{j-1} g_j^2 g_{j-1})^2, g_i^2 g_j g_j^2 g_i, (g_i^2 g_{j+1} g_j^2 g_{j+1})^2,$$
$$\ldots, (g_i^2 g_n g_j^2 g_n)^2\}.$$

The factors occupying the first, second, $i^{\underline{th}}$, and $j^{\underline{th}}$ positions are e by (α). It follows from (α) and (β) that $(g_i^2 g_j^2)^2 = g_i^2 g_j^2 g_i^2 g_j^2 = g_j g_i g_j g_i = e$. It remains to show that $(g_i^2 g_k g_j^2 g_k)^2 = e$ for $k = 4, \ldots, i-1, i+1, \ldots, j-1, j+1, \ldots, n$. Before computing we note that $g_i g_k g_i g_k = e$ implies that $g_k g_i g_k^2 = g_i^2 g_k$. When we use this with those relations already mentioned we have $(g_i^2 g_k g_j^2 g_k)^2 = g_i^2 g_k g_j^2 g_k g_i^2 g_k g_j^2 g_k =$ $g_k g_i g_k^2 g_j^2 g_k g_k g_i g_k^2 g_j^2 g_k = g_k g_i g_j (g_k g_k g_k) g_i g_j g_k g_k =$ $g_k (g_i g_j g_i g_j) g_k g_k = ee = e$.

The elements $\overset{\circ}{t}_i$, $i = 3, \ldots, n$, are distinct since they have different permutations. It is also clear that none of the $\overset{\circ}{t}_i$, $i = 3, \ldots, n$, are E since the permutations are not the identity. Thus we have n-2 elements which generate a group that is the homomorphic image of A(n, n+1).

If $n \geq 5$ the group $T_\circ$ generated is isomorphic to A(n, n+1) by $s\theta = vs = t$ since A(n, n+1) is simple for $n \geq 5$. It is also clear that $V \cap T_\circ = E$ and $V \cup T_\circ = \Sigma$. For if $V \cap T_\circ \neq E$ then there exists an element of the form $v = t = v_1 s \neq E$. This requires that $s = I$ and t is a multiplication. But the permutation part of every t is different from I except for the element of E. This is due to the fact that the isomorphism θ makes correspond to every s of A(n,n+1) an element t with s as its permutation part.

Furthermore, if $V \cup T_0 \neq \Sigma$ there exists an element of the form $y = vs$ which cannot be written as $v_1 t$ where v_1 belongs to V and t belongs to T_0. But $se = v_2 s = t$ is defined, so we can write $y = vs = vv_2^{-1}v_2 s = v_1 t$. Therefore, it follows that $\Sigma_A(H;\ n,\ n+1,\ n+1) = V(n,\ n+1) \cup T_0$, $V(n,\ n+1) \cap T_0 = E$ for $n \geq 5$.

If $n = 4$ the elements $\overset{\circ}{t}_3$, $\overset{\circ}{t}_4$ generate a group that is the homomorphic image of $A(4,5)$. Since $\overset{\circ}{t}_3 = v(1,3,2)$, $\overset{\circ}{t}_4 = v_1(1,4,2)$ the group generated by these elements will have twelve distinct elements. For $\overset{\circ}{t}_3$, $\overset{\circ}{t}_4$, $(\overset{\circ}{t}_3)^2$, $(\overset{\circ}{t}_4)^2$, $(\overset{\circ}{t}_3\overset{\circ}{t}_4)$, $\overset{\circ}{t}_3(\overset{\circ}{t}_4)^2$, $\overset{\circ}{t}_4(\overset{\circ}{t}_3)^2$, $(\overset{\circ}{t}_3)^2\overset{\circ}{t}_4$, $(\overset{\circ}{t}_4)^2\overset{\circ}{t}_3$, $\overset{\circ}{t}_3(\overset{\circ}{t}_4)^2\overset{\circ}{t}_3$, $\overset{\circ}{t}_3\overset{\circ}{t}_4$, $\overset{\circ}{t}_4\overset{\circ}{t}_3$, $(\overset{\circ}{t}_3)^3$ each have different permutations. By the method used when $n \geq 5$ it can be shown that $\Sigma_A(H;\ 4,5,5) = V(4,5) \cup T_0$, $V(4,5) \cap T_0 = E$.

If $n = 3$ the element $\overset{\circ}{t}_3 = \{e,\ e,\ e\}(1,3,2)$ generates a group isomorphic to $A(3,4)$. Similarly, it can be shown that $\Sigma_A(H;\ 3,4,4) = V(3,4) \cup T_0$, $V \cap T_0 = E$.

Corollary 1: The group $\Sigma_A(H;\ 3,4,4)$ splits regularly over its basis group.

Proof: Since $s_3 = (1,3,2)$ generates $A(3,4)$ and $\overset{\circ}{t}_3 = \{e,\ e,\ e\}(1,3,2) = s_3$ generates T_0, it follows that T_0 is $A(3,4)$.

Theorem 1 describes the splitting of the group $\Sigma_A(H;\ 4,5,5)$; however, it is interesting to note that another and perhaps more pleasing construction for the group $T_\circ$ can be given by a slightly different approach.

The group $\Sigma_A(H;\ 4,\ 5,\ 5)$ splits over the basis group, $\Sigma_A(H;\ 4,5,5) = V(4,5) \cup A(4,5)$, $V(4,5) \cap A(4,5) = E$. We shall assume that there exists another group T such that $\Sigma_A = V \cup T$, $V \cap T = E$. The group T is isomorphic to $A(4,5)$ and if we denote the natural isomorphism by θ we have $s\theta = t = vs$ for all s of $A(4,5)$ and some v of $V(4,5)$.

Instead of discussing the images of the elements $s_i = (1,i,2)$ at this point we shall investigate $s_2 = (1,2)(3,4)$, $s_3 = (1,3)(2,4)$, and $s_4 = (1,4)(2,3)$. We shall write $s_2\theta = t_2 = v_2s_2$, $s_3\theta = t_3 = v_3s_3$ and $s_4\theta = t_4 = v_4s_4$. If T is transformed by a multiplication v picked in the proper way, the first factors of v_2, v_3, and v_4 become e. We then work with the group $T_\circ = vTv^{-1}$ and have $s_2\theta = \mathring{v}_2s_2 = \mathring{t}_2$, $s_3\theta = \mathring{v}_3s_3 = \mathring{t}_3$, and $s_4\theta = \mathring{v}_4s_4 = \mathring{t}_4$ where the first factors of $\mathring{v}_2$, $\mathring{v}_3$, $\mathring{v}_4$ are e.

The elements s_i, $i = 2,\ 3,\ 4$, have the property that $(s_i)^2 = I$; so $(\mathring{t}_i)^2 = E$, for $i = 2,\ 3,\ 4$. When these computations are made, one sees that the elements $\mathring{t}_i$ may be written as follows:

$$\mathring{t}_2 = \mathring{v}_2s_2 = \{e,\ e,\ g_{3,2},\ g_{4,2}\}s_2,$$

$$\mathring{t}_3 = \mathring{v}_3s_3 = \{e,\ g_{2,3},\ e,\ g_{4,3}\}s_3,$$

$$\mathring{t}_4 = \mathring{v}_4s_4 = \{e,\ g_{2,4},\ g_{3,4},\ e\}s_4.$$

In $A(4,5)$ $(s_2 s_3) = s_4$; so in T. $\mathring{t}_2\mathring{t}_3 = \mathring{t}_4$. When this is computed and the relations between factors found here combined with those given by $(\mathring{t}_i)^2 = E$, one sees that $s_2\theta = s_2$, $s_3\theta = s_3$, and $s_4\theta = s_4$.

We shall denote $(1,2)(1,4)$ by s_5 and $(1,3)(1,4)$ by s_6. Then $s_5\theta = \mathring{t}_5 = \mathring{v}_5 s_5$ and $s_6\theta = \mathring{t}_6 = \mathring{v}_6 s_6$ for some $\mathring{v}_5$, $\mathring{v}_6$ of the basis group. In the group $A(4,5)$ the following hold: $s_5 s_6 = s_2$, $s_6 s_5 = s_3$, and $(s_5)^3 = (s_6)^3 = I$. The corresponding relations which must hold for elements of T. are $\mathring{t}_5\mathring{t}_6 = s_2$, $\mathring{t}_6\mathring{t}_5 = s_3$, and $(\mathring{t}_5)^3 = (\mathring{t}_6)^3 = E$. When these are computed and the relations that must hold on the factors of $\mathring{v}_5$, $\mathring{v}_6$ found, we see that $\mathring{t}_5 = \{g\}s_5$, $\mathring{t}_6 = \{g^2\}s_6$. Further, $g^3 = e$.

The other elements of $A(4,5)$ are $s_1 = I$, $s_7 = (1,4)(1,2)$, $s_8 = (1,3)(1,2)$, $s_9 = (2,4)(2,3)$, $s_{10} = (2,3)(2,4)$, $s_{11} = (1,2)(1,3)$, and $s_{12} = (1,4)(1,3)$. The last six may be expressed as follows: $s_7 = s_6 s_2$, $s_8 = s_5 s_4$, $s_9 = s_6 s_3$, $s_{10} = s_5 s_2$, $s_{11} = s_6 s_4$, $s_{12} = s_5 s_3$. That is, they are expressed in terms of permutations whose images under θ are known. It follows that $s_7\theta = \{g^2\}s_7$, $s_8\theta = \{g\}s_8$, $s_9\theta = \{g^2\}s_9$, $s_{10}\theta = \{g\}s_{10}$, $s_{11}\theta = \{g^2\}s_{11}$, $s_{12}\theta = \{g\}s_{12}$.

Since scalars commute with permutations, we see that the group G generated by g is the homomorphic image of $A(4,5)$ by the correspondence $s\phi = g_s$ where g_s is the factor in the scalar of $s\theta$. It is also clear that the subgroup of $A(4,5)$ whose elements are s_1, s_2, s_3, s_4 is contained in the kernel of the homomorphism.

We are led to the following theorem:

<u>Theorem 2</u>: The group $\Sigma_A(H;4,5,5)$ splits over the basis group, $\Sigma_A(H;4,5,5) = V(4,5) \cup T$, $V(4,5) \cap T = E$. Any group T in this decomposition is the conjugate to some group $T_\circ$ obtained as follows. Let G be a subgroup of H which is the identity or a cyclic group of order three. Let g be the generator of G. G is the homomorphic image of $A(4,5)$, and we denote this homomorphism by $s\phi = g_s$ for s in $A(4,5)$. The elements of $T_\circ$ are obtained from those of $A(4,5)$ by the isomorphism

$$s\theta = \{g_s\}s$$

for s belonging to $A(4,5)$.

It has already been shown that any group T, after suitable transformation, must have the form described in the theorem. It remains to show that θ is an isomorphism. It is clear that θ is one-to-one and defined.

That multiplication is preserved follows immediately from the fact that scalars commute with permutations and that ϕ is a homomorphism.

By the method used in the proof of Theorem 1 it can be shown that $V(4,5) \cap T_\circ = E$ and $\Sigma_A(H;\ 4,\ 5,\ 5)$ $= V(4,5) \cup T_\circ$.

Before proceeding we mention one other special construction. The group $\Sigma_A(H;\ 5,\ 6,\ 6)$ splits over the basis group. If we assume that there exists a group T such that $\Sigma_A(H;\ 5,\ 6,\ 6) = V(5,6) \cup T$, $V(5,6) \cap T = E$, then T is isomorphic to $A(5,6)$ and we denote the natural isomorphism by θ where $s\theta = t = vs$ for some v of the basis group. In the group $A(5,6)$ we shall use the notation $s_1 = I$, $s_2 = (1,2,3,4,5)$, $s_3 = (1,3,5,2,4)$, $s_4 = (1,4,2,5,3)$, and s_5, $(1,5,4,3,2)$. We have that $s_1\theta = E$, $s_2\theta = t_2 = v_2s_2$, $s_3 = t_3 = v_3s_3$, $s_4 = t_4 = v_4s_4$, $s_5\theta = t_5 = v_5s_5$. We transform T by a multiplication v in such a way that $T_\circ = vTv^{-1}$ has elements $\mathring{t}_2$, $\mathring{t}_3$, $\mathring{t}_4$, $\mathring{t}_5$ whose first factors are e. We then use the relations $s_3 = (s_2)^2$, $s_4 = (s_2)^3$, $s_5 = (s_2)^4$, $s_1 = (s_2)^5$ in order to find relations on the factors of $\mathring{t}_i$, $i = 2, \ldots, 5$. A few computations show that $s_i\theta = s_i$ for $i = 1, \ldots, 5$.

In $A(5,6)$ if we denote $(1,4)(1,2)$ by s_6, $(2,3)(4,5)$ by s_7, and $(1,2)(1,4)$ by s_8 we have that $s_6s_7 = s_5$, $(s_6)^3 = I$, $(s_7)^2 = I$, $s_6s_8 = I$, $s_7s_8 = s_2$. The corresponding relations for elements of $T_\circ$ are $\mathring{t}_6\mathring{t}_7 = \mathring{t}_5$, $(\mathring{t}_6)^3 = E$, $(\mathring{t}_7)^2 = E$, $\mathring{t}_6\mathring{t}_8 = E$, $\mathring{t}_7\mathring{t}_8 = \mathring{t}_2$ where $s_6\theta = \mathring{t}_6$, $s_7\theta = \mathring{t}_7$, $s_8\theta = \mathring{t}_8$. Another set of computations leads to the following result. The factors of $\mathring{t}_6$, $\mathring{t}_7$, and $\mathring{t}_8$ can be expressed in terms of two elements, g_3, g_4 say, of H which must satisfy $(g_3)^3 = e$, $(g_4)^3 = e$, and $(g_3g_4)^2 = e$.

$s_6 e = \{g_4^2, g_3g_4, g_3, g_3^2, g_4\}s_6$

$s_7 e = \{g_3g_4, g_3, g_3^2, g_4, g_4^2\}s_7$

$s_8 e = \{g_3g_4, g_3, g_3^2, g_4, g_4^2\}s_8$

All of the elements of A(5,6) can be expressed in terms of the eight elements of A(5,6) which we have listed. Therefore, all of the factors of images of elements of A(5,6) can be expressed in terms of g_3 and g_4. It is a somewhat lengthy computation to find these factors and the details of the work and the results will not be listed. However, we are led to the following theorem:

<u>Theorem 3</u>: The group $\Sigma_A(H; 5, 6, 6)$ splits over the basis group, $\Sigma_A(H; 5, 6, 6) = V(5,6) \cup T$, $V(5,6) \cap T = E$. Any group T of this decomposition is the conjugate of some group $T_\circ$ constructed as follows. Let G be a subgroup of H that is the homomorphic image of A(4,5). Let g_3, g_4 be generators of G, satisfying the relations $(g_i)^3 = e$ for $i = 3, 4$ and $(g_3g_4)^3 = e$. Then the elements of $T_\circ$ are obtained from those of A(5,6) by the isomorphism

$(1,4,5)e = \overset{\circ}{t}_2 = \{g_4g_3, g_4g_3^2, g_4^2g_3, g_3g_4^2, g_3^2g_4\}(1,4,5)$,

$(2,4,5)e = \overset{\circ}{t}_3 = \{g_3^2, g_4, g_4^2, g_3, g_4, g_3\}(2,4,5)$,

$(3,4,5)e = \overset{\circ}{t}_4 = \{g_4g_3^2, g_4^2g_3, g_3g_4^2, g_3^2g_4, g_4g_3\}(3,4,5)$.

By the procedure used in the proof of Theorem 1 we can show that $\overset{\circ}{t}_2$, $\overset{\circ}{t}_3$, $\overset{\circ}{t}_4$ satisfy $(\overset{\circ}{t}_1)^3 = E$ for $i = 2, 3, 4$ and $(\overset{\circ}{t}_i\overset{\circ}{t}_j)^2 = E$ for $i \neq j$. Hence the three distinct non-identity elements generate a group $T_\circ$ which is the homomorphic image of $A(5,6)$. But $A(5,6)$ is simple so the group $T_\circ$ is isomorphic to $A(5,6)$. Furthermore, by the previous method it follows that $V(5,6) \cap T_\circ = E$ and $\Sigma_A(H;\ 5,\ 6,\ 6) = V(5,6) \cup T_\circ$.

We return now to the general case where $\Sigma_A(H;\ n,\ n+1,\ n+1)$ is the group under discussion. We shall show as an immediate result of Theorem 1 the following:

<u>Theorem 4</u>: A necessary and sufficient condition for the group $\Sigma_A(H;\ n,\ n+1,\ n+1)$ to split regularly over the basis group is that H contain no subgroup, except e, which is the homomorphic image of $A(n-1,\ n)$.

<u>Proof</u>: If H contain no subgroup, except e, which is homomorphic to $A(n-1,\ n)$, then the group $T_\circ$ constructed in Theorem 1 is simply $A(n,\ n+1)$.

Conversely, if Σ_A splits regularly, then $T_\circ$ can be transformed into $A(n,\ n+1)$. By the method used in Theorem 2, Section 1, Chapter III, it can be shown that such a transformation need only be by a multiplication. Let $v = \{k_1,\ k_2,\ \ldots,\ k_n\}$ be this multiplication. Consider the element $v\overset{\circ}{t}_3v^{-1}$.

$vt_3^{\circ}v^{-1} =$

$\{k_1k_3^{-1}, k_2k_1^{-1}, k_3k_2^{-1}, k_4g_4k_4^{-1}, \ldots, k_ng_nk_n^{-1}\}(1,3,2)$

If $vt_3^{\circ}v^{-1}$ is a permutation, then $k_ig_ik_i^{-1} = e$ for $i = 4, \ldots, n$. But this means that $g_i = e$ for $i = 4, \ldots, n$ and the group G is simply e.

<u>Corollary 1</u>: A necessary and sufficient condition for $\Sigma_A(H; n, n+1, n+1)$, for $n = 4,5$ to split regularly over the basis group is that H contain no element of order 3.

<u>Proof</u>: If H contains no element of order 3, then H contains no subgroup, except e, which is the homomorphic image of $A(n-1, n)$. Hence, by Theorem 4, $\Sigma_A(H; n, n+1, n+1)$ splits regularly.

Conversely, if $\Sigma_A(H; n, n+1, n+1)$ splits regularly, by Theorem 4, H contains no subgroup, except e, which is the homorphic image of $A(n-1, n)$. Therefore, H contains no element of order 3.

<u>Corollary 2</u>: A necessary and sufficient condition for $\Sigma_A(H; n, n+1, n+1)$, for $n \geq 6$, to split regularly over the basis group is that H contain no subgroup isomorphic to $A(n-1, n)$.

<u>Proof</u>: If H contains no subgroup isomorphic to $A(n-1, n)$, then H contains no subgroup, except e, which is the homomorphic image of $A(n-1, n)$, since $A(n-1, n)$ is simple.

Conversely, if $\Sigma_A(H;\ n,\ n+1,\ n+1)$ splits regularly, then H contains no subgroup, except e, which is the homomorphic image of $A(n-1,\ n)$. This means that H contains no subgroup isomorphic to $A(n-1,\ n)$.

3. <u>The Splitting of $\Sigma_A(H;\ B,\ B^+,\ d)$</u>

It has been shown that $\Sigma_A(H;\ B,\ B^+,\ d)$ splits over the basis group, $\Sigma_A(H;\ B,\ B^+,\ d) = V(B,B^+) \cup A(B,d)$, $V(B,B^+) \cap A(B,d) = E$. We shall consider the problem of finding all groups T such that $\Sigma_A(H;\ B,\ B^+,\ d) = V(B,B^+) \cup T$, $V(B,B^+) \cap T = E$.

Let us assume that there exists such a group T. We see that T is isomorphic to $A(B,d)$ and that the isomorphism θ can be taken such that $s\theta = t = vs$.

In a manner similar to that used to show that $s_i = (1,i,2)$, $i = 3, \ldots, n$, generate $A(n,\ n+1)$ it can be shown that $s_\alpha = (1,\alpha,2)$ generate a group that is the homomorphic image of $A(B,d)$. Since, by Theorem 8 of Chapter I, $A(B,d)$ is simple, the group generated is $A(B,d)$. T must contain the elements $s_\alpha\theta = t_\alpha = v_\alpha s_\alpha$ for some v_α of $V(B,B^+)$ and all α. We shall denote t_α as

$$t_\alpha = \{h_{1,\alpha} h_{2,\alpha},\ \ldots,\ h_{\varepsilon,\alpha},\ \ldots\} s_\alpha.$$

The method used for constructing conjugates of T in the previous section for $\Sigma_A(H;\ n,\ n+1,\ n+1)$ will again be used. There exists an element v of the basis group such that $T_\circ = vTv^{-1}$ must contain $vt_\alpha v^{-1} = t^\circ_\alpha$

whose first factors are e for all α and the second factor of $\overset{\circ}{t}_3$ is also e. We now work with $T_\circ$ and show that the factors of $\overset{\circ}{t}_\alpha$ for $\alpha > 3$ can be expressed in terms of the factors of $\overset{\circ}{t}_3$. The factors of $\overset{\circ}{t}_3$ will be denoted by $g_{\alpha,3}$ where $\alpha = 3, 4, \ldots$.

$$\overset{\circ}{t}_3 = \{e, e, g_{3,3}, \ldots, g_{\varepsilon,3}, \ldots\}(1,3,2).$$

Since $(s_\alpha)^3 = I$ we must have $(\overset{\circ}{t}_\alpha)^3 = E$, and we gain the relations

(1) $g_{3,3} = e$,

(2) $g^3_{\varepsilon,3} = e$ for $\varepsilon = 4, 5, \ldots$,

(3) $g_{\alpha,\alpha}g_{2,\alpha} = e$ for $\alpha = 4, 5, \ldots$.

Since $(s_3s_\alpha)^2 = I$ for $\alpha = 4, 5, \ldots$, we must have $(\overset{\circ}{t}_3\overset{\circ}{t}_\alpha)^2 = E$. This computation yields the relations

(4) $g_{3,\alpha}g_{2,\alpha} = e$ for $\alpha = 4, 5, \ldots$,

(5) $g_{\alpha,3}g_{\alpha,\alpha} = e$ for $\alpha = 4, 5, \ldots$,

(6) $(g_{\varepsilon,3}g_{\varepsilon,\alpha})^2 = e$ for all ε except $1,2,3,\alpha$ and for all $\alpha = 4, 5, \ldots$.

A combination of (5) and (2) gives (7); (7), (3), and (2) give (8), and (4), (2), and (8) give (9).

(7) $g_{\alpha,\alpha} = g^2_{\alpha,3}$ for $\alpha = 4, 5, \ldots$,

(8) $g_{2,\alpha} = g_{\alpha,3}$ for $\alpha = 4, 5, \ldots$,

(9) $g_{3,\alpha} = g^2_{\alpha,3}$ for $\alpha = 4, 5, \ldots$.

This means that

$$\overset{\circ}{t}_3 = \{e, e, e, g_{4,3}, \ldots, g_{\varepsilon,3}, \ldots\}(1,3,2)$$

$$\overset{\circ}{t}_\alpha = \{e, g_{\alpha,3}, g^2_{\alpha,3}, g_{4,\alpha}, \ldots, g^2_{\alpha,3}, \ldots, g_{\varepsilon,\alpha}, \ldots\}(1,\alpha,2)$$

where $g^2_{\alpha,3}$ appears in the third and $\alpha^{\underline{th}}$ positions and $\alpha = 4, 5, \ldots$.

In order to express the factors of $\overset{\circ}{t}_\alpha$ in terms of the factors of $\overset{\circ}{t}_3$ we note that $(s_\alpha s_\beta)^2 = I$, for $\alpha \neq \beta$, in $A(B,d)$ so $(\overset{\circ}{t}_\alpha \overset{\circ}{t}_\beta)^2 = E$ in $T_\circ$. When $(\overset{\circ}{t}_\alpha \overset{\circ}{t}_\beta)^2$ is computed we find

$$(\overset{\circ}{t}_\alpha \overset{\circ}{t}_\beta)^2 = \{g_{\alpha,\beta} g^2_{\alpha,3} g_{\beta,3}, \ldots\}$$

which implies that

$$(10) \quad g_{\alpha,\beta} g^2_{\alpha,3} g_{\beta,3} = e$$

or when (2) is used that

$$(11) \quad g_{\alpha,\beta} = g^2_{\beta,3} g_{\alpha,3}.$$

We can now rewrite $\overset{\circ}{t}_\alpha$ as follows:

$$\overset{\circ}{t}_\alpha = \{e, g_{\alpha,3}, g^2_{\alpha,3}, \ldots, g^2_{\alpha,3} g_{\varepsilon,3}, \ldots, g^2_{\alpha,3}, \ldots\}(1,\alpha,2)$$

where $g^2_{\alpha,3} g_{\varepsilon,3}$ appears in the $\varepsilon^{\underline{th}}$ position for $\varepsilon \neq 1,2,3,\alpha$ and $g^2_{\alpha,3}$ appears in the third and $\alpha^{\underline{th}}$ positions and $\alpha > 3$.

The set of elements $g_{\varepsilon,3}$, where $\varepsilon = 4, \ldots$, generate a group which is the homomorphic image of $A(B,d)$. For $(g_{\varepsilon,3})^3 = e$ by (2), and we can show $(g_{\varepsilon,3} g_{\delta,3})^2 = e$ for all ε and δ such that $\varepsilon \neq \delta$ by use of (6), (11), and (2) as follows:

$$(g_{\varepsilon,3} g_{\varepsilon,\delta})^2 = e$$

$$(g_{\varepsilon,3} g^2_{\delta,3} g_{\varepsilon,3})^2 = g_{\varepsilon,3} g^2_{\delta,3} g^2_{\varepsilon,3} g^2_{\delta,3} g_{\varepsilon,3} = e$$

$$g^2_{\delta,3} g^2_{\varepsilon,3} g^2_{\delta,3} = g^4_{\varepsilon,3} = g_{\varepsilon,3}$$

$g^2_{\varepsilon,3} = g_{\delta,3}g_{\varepsilon,3}g_{\delta,3}$

$e = g_{\delta,3}g_{\varepsilon,3}g_{\delta,3}g_{\varepsilon,3}$.

We are led to the following theorem:

Theorem 1: The group $\Sigma_A(H;\, B,\, B^+,\, d)$ splits over the basis group, $\Sigma_A(H;\, B,\, B^+,\, d) = V(B,B^+) \cup T$, $V(B,B^+) \cap T = E$. The group T is conjugate to some group $T_\circ$ obtained as follows. Let G be a subgroup of H which is the homomorphic image of A(B,d). Let $g_4, \ldots, g_\varepsilon, \ldots$ be generators of G, satisfying the relations (i) $(g_\varepsilon)^3 = e$ and (ii) $(g_\varepsilon g_\delta)^2 = e$ when $\varepsilon \neq \delta$. Let $s_\alpha = (1,\alpha,2)$, $\alpha = 3, \ldots,$ denote the generators of the group A(B,d). Then the elements of $T_\circ$ are obtained from the elements of A(B,d) by the isomorphism θ defined by

$$s_3\theta = \overset{\circ}{t}_3 = \{e,e,e,g_4,\ldots,g_\varepsilon,\ldots\}(1,3,2)$$

$$s_\alpha\theta = \overset{\circ}{t}_\alpha = \{e,g_\alpha,g^2_\alpha,g^2_\alpha g_4,\ldots,g^2_\alpha,\ldots,\;g^2_\alpha g_\varepsilon,\ldots\}(1,\alpha,2).$$

It has already been shown that any group T after suitable transformation must have the form indicated. We shall now show that $(\overset{\circ}{t}_\alpha)^3 = E$ for $\alpha = 3, \ldots$ and $(\overset{\circ}{t}_\alpha\overset{\circ}{t}_\beta)^2 = E$ for $\alpha \neq \beta$. $(\overset{\circ}{t}_3)^3 = E$ follows from (i). $(\overset{\circ}{t}_\alpha)^3 = vs$ has $s = I$ and v has e factors in the first, second, third and αth positions by (i). It remains to show $(g^2_\alpha g_\varepsilon)^3 = e$ for ε different from 1, 2, 3, α.

This follows immediately from (i) and (ii). We now show $(\overset{\circ}{t}_\alpha \overset{\circ}{t}_\beta)^2 = E$ for all α and β such that $\alpha \neq \beta$.

If one considers first $(\overset{\circ}{t}_3 \overset{\circ}{t}_\beta)^2 = vs$, one sees that $s = I$ and v has e for factors in the first, second, third, and $\beta^{\underline{th}}$ positions by (i). $(\overset{\circ}{t}_3 \overset{\circ}{t}_\beta)^2 = E$ then if $(g_\varepsilon g_\beta^2 g_\varepsilon)^2 = e$ and this follows from (i) and (ii). Next $(\overset{\circ}{t}_\alpha \overset{\circ}{t}_3)^2 = vs$ where $s = I$ and v has factors of e in the first, second, third, and $\alpha^{\underline{th}}$ positions. When ε is different from 1, 2, 3, α, we have the $\varepsilon^{\underline{th}}$ factor of v is $(g_\alpha^2 g_\varepsilon g_\varepsilon)^2 = g_\alpha^2 g_\varepsilon^2 g_\alpha^2 g_\varepsilon^2 = g_\varepsilon g_\alpha g_\varepsilon g_\alpha = e$. Finally, we show $(\overset{\circ}{t}_\alpha \overset{\circ}{t}_\beta)^2 = vs = E$ when $\alpha \neq 3$, $\beta \neq 3$, $\alpha \neq \beta$. Again $s = I$ and v has factors of e in the first, second, $\alpha^{\underline{th}}$, and $\beta^{\underline{th}}$ positions. The general factor is $(g_\alpha^2 g_\varepsilon g_\beta^2 g_\varepsilon)^2$ for $\varepsilon \neq 1, 2, 3, \alpha, \beta$. This may be shown to be e by (i) and (ii). The third factor is $(g_\alpha^2 g_\beta^2)^2$ which is e by (i) and (ii).

It follows that we have B distinct non-identity elements $\overset{\circ}{t}_\alpha$, $\alpha = 3, \ldots,$ with the properties that $(\overset{\circ}{t}_\alpha)^3 = E$ for $\alpha = 3, \ldots,$ and $(\overset{\circ}{t}_\alpha \overset{\circ}{t}_\beta)^2 = E$ for $\alpha \neq \beta$. These elements generate a group $T_\circ$ which is the homomorphic image of $A(B,d)$. Since $A(B,d)$ is simple the group $T_\circ$ is isomorphic to $A(B,d)$. By the procedure used in the proof of Theorem 1 we can show that $V(B,B^+) \cap T_\circ = E$ and $\Sigma_A(H;\, B,\, B^+,\, d) = V(B,B^+) \cup T_\circ$.

The last theorem leads to the following:

Theorem 2: A necessary and sufficient condition for the group $\Sigma_A(H;\ B,\ B^+,\ d)$ to split regularly over the basis group is that H contain no subgroup different from e that is the homomorphic image of $A(B,d)$.

The proof is the same as that for Theorem 4 of Section 2 of this chapter.

Corollary 1: A necessary and sufficient condition for $\Sigma_A(H;\ B,\ B^+,\ d)$ to split regularly over the basis group is that H contain no subgroup isomorphic to $A(B,d)$.

Proof: If H contains no subgroup isomorphic to $A(B,d)$ then, by Theorem 8 of Chapter I, H contains no subgroup, except e, which is the homomorphic image of $A(B,d)$. Therefore, by Theorem 2, $\Sigma_A(H;\ B,\ B^+,\ d)$ splits regularly over the basis group.

Conversely, if $\Sigma_A(H;\ B,\ B^+,\ d)$ splits regularly over the basis group, by Theorem 2, H contains no subgroup, except e, which is the homomorphic image of $A(B,d)$. Therefore, H contains no subgroup isomorphic to $A(B,d)$.

Corollary 2: For every group H there exists a group $\Sigma_A(H;\ B,\ B^+,\ d)$ such that the monomial group splits regularly over the basis group.

Proof: This follows from Corollary 1 if the cardinal B is chosen such that $o(A(B,d)) > o(H)$.

CHAPTER IV

Normal Subgroups of the Symmetry

In this chapter all of the normal subgroups of the groups $\Sigma(H;\ B,\ d,\ d)$, $\Sigma_A(H;\ B,\ d,\ d)$, $\Sigma_A(H;\ n,\ n+1,\ n+1)$ for $n \geqq 5$ and $\Sigma_A(H;\ 2,\ 3,\ 3)$ are found. The method of investigation is that employed by Ore [1] for $\Sigma(H;\ n,\ n+1,\ n+1)$.

1. Normal Subgroups of $\Sigma(H;\ B,\ d,\ d)$

Before the normal subgroups of the symmetry can be determined we must solve a preliminary problem.

Definition: *A subgroup of the symmetry $\Sigma(H;\ B,\ d,\ d)$ is permutation invariant if it is transformed into itself by all permutations s of the symmetric group $S(B,d)$.*

The first problem to be solved is that of finding all permutation invariant subgroups of $\Sigma(H;\ B,\ d,\ d)$ contained in the basis group $V(B,d)$.

1.1 Permutation Invariant Subgroups of $\Sigma(H;\ B,\ d,\ d)$ Contained in the Basis Group

Let N be a fixed permutation invariant subgroup of Σ contained in the basis group. Let v be an arbitrary element of N. Since all elements of the basis group have only a finite number of factors different from e, v has only a finite number of non-identity factors. All elements g_α of H which occur in the $\alpha^{\underline{th}}$ position of multiplications in N will form a subgroup of H. Since N is permutation invariant, this subgroup will be the same for all indices α, so we denote it by G.

The set S_1 of all the multiplications in N that have every factor $h_\alpha = e$ for $\alpha > 1$ form a subgroup of N. When S_1 is transformed by any multiplication v of N the result is S_1. This shows that S_1 is a normal subgroup of N.

The factors that occur in the first position of multiplications of S_1 form a subgroup G_1 of G, and G_1 is normal in G.

Let $v = \{g_1, e, \ldots, e, \ldots\}$ be any element of S_1. Since N is permutation invariant, N must contain the element $(1,\alpha)v(1,\alpha) = v_\alpha = \{e, e, \ldots, e, g_1, e, \ldots\}$ where g_1 appears in the $\alpha^{\underline{th}}$ position. The set of multiplications that have e as a factor in every position different from α form a subgroup S^α of N. It is clear that the factors in the $\alpha^{\underline{th}}$ position of elements of S^α form a subgroup of G which is the same as G_1 and that S^α is also normal in N. Furthermore, $S^\alpha \cap S^\beta = E$ for $\alpha \neq \beta$. Let W be the weak direct product of the set of normal subgroups S^α. W is itself a normal subgroup of N.

The preceding shows that if v is an arbitrary element of N then any of the non-identity factors of v can be multiplied by an arbitrary g of G_1 and the multiplication v_o so obtained will again be in N. Thus the relations between the factors of v can only be modulo G_1. It is, therefore, no limitation if we consider the quotient group G/G_1 in the following and <u>assume $G_1 = e$</u> for the proofs of Theorems 1, 2, 3, and 4.

The set of elements S_2 of N which have every factor $h_\alpha = e$ for $\alpha > 2$ form a subgroup of N. The first factors of elements of S_2 run through a subgroup G_2 of G. G_2 is the same group as the group that the second factors of elements of S_2 run through since N is permutation invariant. It is easily verified that S_2 is normal in N and G_2 is normal in G.

We may, therefore, apply the result of Ore [1 Theorem 1, p. 29] in regard to groups of the form of S_2 and conclude that elements of S_2 have the form

$$v_2 = \{g_2, g_2^T, e, \ldots, e, \ldots\}$$

where T is some automorphism of order two of the group G_2.

Since N is permutation invariant we see that, if v is an element of N that has h_α and h_β as α<u>th</u> and β<u>th</u> factors with the remaining factors e, then h_α, h_β are elements of G_2 and $h_\alpha = h_\beta^T$.

When N contains $v_1 = \{g_2, g_2^T, e, \ldots, e, \ldots\}$, it also contains $v_2 = (2,3)v_1(2,3) = \{g_2, e, g_2^T, \ldots, e, \ldots\}$. Furthermore, N contains $v_3 = v_1 v_2^{-1} = \{e, g_2^T, (g_2^T)^{-1}, \ldots e, \ldots\}$. Again using the fact that N is permutation invariant we see N must contain

$$v_4 = (1,2,3)v_3(1,3,2) = \{g_2^T, (g_2^T)^{-1}, e, \ldots, e, \ldots\}.$$

This shows that $g_2^T = g_2^{-1}$. But if a group has an automorphism changing every element into its inverse the group is Abelian. We have shown the following:

<u>Theorem 1</u>: Let N be a fixed permutation invariant subgroup of $\Sigma(H; B, d, d)$ contained in the basis group. Then the set G of H consisting of all of the factors that occur in a fixed $\alpha^{\underline{th}}$ position of all the multiplications in N form a subgroup of H. This group is the same for all α. The set S_1 of all multiplications of N which have $h_\alpha = e$ for $\alpha > 1$ form a normal subgroup of N. The set G_1 consisting of first factors of multiplications of S_1 form a normal subgroup of G. We shall assume $G_1 = e$. The set of elements S_2 of N that have $h_\alpha = e$ for $\alpha > 2$ form a normal subgroup of N. The set G_2 of first factors of elements of S_2 form a normal, Abelian subgroup of G. The elements of S_2 are of the form

$$v_2 = \{g_2, g_2^{-1}, e, \ldots, e, \ldots\}$$

where g_2 runs through G_2.

When the factors g_2, g_2^{-1} in v_2 are permuted into all possible positions, the corresponding substitutions generate a normal subgroup R of N which is also permutation invariant. From the construction of R it follows that if v is one of its elements and the non-identity factors of v are $r_{i_1}, r_{i_2}, \ldots, r_{i_n}$ then $r_{i_1} r_{i_2} \ldots r_{i_n} = e$.

R must contain the elements

$v_1 = \{e, \ldots, e, p_1, e, \ldots, e, p_1^{-1}, e, \ldots\}$

$v_2 = \{e, \ldots, e, p_2, e, \ldots, e, p_2^{-1}, e, \ldots\}$

— —

$v_{n-1} = \{e, \ldots, e, p_{n-1}, e, \ldots, e, p_{n-1}^{-1}, e, \ldots\}$

where p_i^{-1} occurs in the same position as p_{i+1} for $i = 1, \ldots, n-2$ and the p_i are arbitrary in G_2. Then R must contain

$v_1 v_2 \ldots v_{n-1} =$

$\{e, \ldots, e, p_1, e, \ldots, e, p_1^{-1}p_2, e, \ldots,$

$e, p_{n-2}^{-1}p_{n-1}, e, \ldots, e, p_{n-1}^{-1}, e, \ldots\}.$

Conversely, every element of R can be expressed in this form. For let $v = \{e, \ldots, e, r_{i_1}, e, \ldots, e, r_{i_n}, e, \ldots\}$ be any element of R and choose $p_1 = r_{i_1}$, $p_2 = r_{i_1}r_{i_2}, \ldots,$ $p_{n-2} = r_{i_1} \ldots r_{i_{n-2}}$, $p_{n-1} = r_{i_1} \ldots r_{i_{n-1}}$. This shows that by a proper choice of p_i for $i = 1, \ldots, n-1$ we can make the first n-1 non-identity factors of an element of R arbitrary in G_2, but the remaining non-identity factor is uniquely determined by $r_{i_1} \ldots r_{i_n} = e$. We have shown the following:

Theorem 2: The group R, generated by the substitutions obtained by all possible permutations of elements of S_2, consists of elements of the form

$$v = \{e, \ldots, e, r_{i_1}, e, \ldots,$$
$$e, r_{i_{n-1}}, e, \ldots,$$
$$e, (r_{i_1} \ldots r_{i_{n-1}})^{-1}, e, \ldots\}$$

where the r_{i_j} run through the Abelian group G_2 independently.

We now come to the final step in the determination of the permutation invariant subgroups of Σ that are contained in the basis group. Let v be an arbitrary element of N and let the non-identity factors occur in the $i_1, \ldots, i_n$ positions. Since N is permutation invariant it must contain v transformed by (i_1,α) where α is different from each of the indices $i_1, \ldots, i_n$. Let $v_1 = (i_1,\alpha)v(i_1,\alpha)$. Then v_1 differs from v only by having the factor g_{i_1} in different positions. We form vv_1^{-1} and the result is

$$\{e, \ldots, e, g_{i_1}, e, \ldots, e, g_{i_1}^{-1}, e, \ldots\}$$

where g_{i_1} is in the i_1<u>th</u> position and $g_{i_1}^{-1}$ is in the α<u>th</u> position. This shows that g_{i_1} belongs to G_2. A similar procedure shows that all the factors of v are in G_2 and this means that N is simply R.

We have shown that if N is permutation invariant and contained in V, with $G_1 = e$, then N must be the group R of Theorem 2.

Conversely, any group R whose elements are of the form

$$v = \{e, \ldots, e, r_{i_1}, e, \ldots, e, r_{i_{n-1}}, e, \ldots, e, (r_{i_1} \ldots r_{i_{n-1}})^{-1}, e, \ldots\}$$

where r_{i_j} runs through an Abelian subgroup G_2 of H is permutation invariant. This follows from the fact that svs^{-1} has non-identity factors in G_2 with the property that the product of the non-identity factors is e.

We have, therefore, obtained the following result:

<u>Theorem 3</u>: Let an Abelian subgroup G be chosen in H. The group N consisting of all elements of the form

$$v = \{e, \ldots, e, g_{i_1}, e, \ldots, e, g_{i_{n-1}}, e \ldots, e, (g_{i_1} \ldots g_{i_{n-1}})^{-1}, e, \ldots\},$$

where the g_{i_j} run through G independently, is a permutation invariant subgroup of $\Sigma(H;\, B,\, d,\, d)$.

We recall that for convenience the subgroup G_1 was assumed to be e. The results obtained may, therefore, be generalized by working with G_1 a normal subgroup of $G = G_2$ and $G_1 \neq e$. Such a consideration leads to a determination of all permutation invariant subgroups of $\Sigma(H;\, B,\, d,\, d)$ contained in the basis group. This result is stated in the following theorem.

<u>Theorem 3'</u>: All permutation invariant subgroups N of $\Sigma(H;\, B,\, d,\, d)$ that are contained in the basis group may be obtained by the following construction. A subgroup G of H

is chosen. In G a normal subgroup G_1 is chosen such that the quotient group G/G_1 is Abelian. Then let N consist of elements of the form

$$v = \{e, \ldots, e, g_{i_1}, e, \ldots, g_{i_n}, e, \ldots\}$$

where the non-identity factors run through G subject to the condition that $g_{i_1} \ldots g_{i_n}$ belongs to G_1. Conversely, any such N is permutation invariant.

1.2 Normal Subgroups of $\Sigma(H;\ B,\ d,\ d)$ Contained in the Basis Group

A necessary and sufficient condition for a subgroup N of the basis group to be normal in Σ is that N be permutation invariant and normal in $V(B,d)$. For if v is an element of N and N satisfies the above conditions, then for any element $y = v_1 s_1$ of Σ we have $yvy^{-1} = v_1 s_1 v s_1^{-1} v_1^{-1}$ $= v_1 v_2 v_1^{-1} = v_3$ where v_2 is an element of N since N is permutation invariant and v_3 is an element of N since N is normal in the basis group. Conversely, if N is normal in Σ, it is normal in the basis group and permutation invariant.

So we shall have to find those normal subgroups of the basis group which are permutation invariant. We shall use the same notation as in the previous section.

Let N be a normal subgroup of Σ contained in $V(B,d)$. Now the group $G = G_2$ must be a normal subgroup of H and the group G_1 is also normal in H. Since N is permutation invariant, an element v of N must have the form described in Theorem $3'$. For convenience let us for the moment again assume that $G_1 = e$. Then any non-identity factor of an element of N is uniquely determined by the other non-identity factors. Let $v_1 = \{e, \ldots, e, h, e, \ldots\}$ be an element of $V(B,d)$ that has h as its only non-identity factor. Furthermore, let h occupy the position occupied by g_{i_1} of v and let h be arbitrary in H. The element

$$v_1 v v_1^{-1} = \{e, \ldots, e, hg_{i_1}h^{-1}, e, \ldots, \\ e, (g_{i_1} \ldots g_{i_{n-1}})^{-1}, e, \ldots\}$$

must be in N since N is normal in $V(B,d)$. This means that $hg_{i_1}h^{-1} = g_{i_1}$, and since this is true for all h of H and all g of G we have that G belongs to the center of H.

Conversely, if $G_1 = e$, $G_2 = G$ belongs to the center of H and N is permutation invariant, then N is normal in $\Sigma(H;\ B,\ d,\ d)$.

We have, therefore, established:

<u>Theorem 4</u>: If N is as given by Theorem 3 and the additional requirement that G belongs to the center of H is met, then H is normal in $\Sigma(H;\ B,\ d,\ d)$.

The assumption $G_1 = e$ was not necessary and Theorem 4 may be generalized by assuming $G_1 \neq e$. The non-identity factors of elements of N are uniquely determined by the other non-identity factors modulo G_1. Then $hg_{i_1}h^{-1} \equiv g_{i_1}$ modulo G_1 and G/G_1 belongs to the center of H/G_1. If $G_1 \subset G$ are normal subgroups of H, G/G_1 belongs to the center of H/G_1 and N is permutation invariant, then N is normal in $\Sigma(H; B, d, d)$. This establishes:

<u>Theorem 4'</u>: The normal subgroups of $\Sigma(H; B, d, d)$ that are contained in the basis group are obtained from the construction of Theorem 3' with the additional conditions that $G_1 \subset G$ are normal in H, and G/G_1 belongs to the center of H/G_1.

1.3 <u>Other Normal Subgroups of $\Sigma(H; B, d, d)$</u>

We shall completely solve the problem of finding all normal subgroups of Σ by finding those normal subgroups which are not contained in the basis group. Let M be such a normal subgroup of Σ. The intersection of M and the basis group will be a normal subgroup of Σ contained in the basis group. That is, $N = M \cap V$ is of the form described in Theorem 4'.

Let y be a substitution of M and let c be a cycle of y, $c = \begin{pmatrix} x_{i_1}, & \ldots, & x_{i_n} \\ h_1 x_{i_2}, & \ldots, & h_n x_{i_1} \end{pmatrix}$. Let $v = \{k_1, k_2, \ldots, k_\varepsilon, \ldots\}$ be an element of the basis group. Since M is normal in Σ, the multiplication defined by the commutator $y^{-1}v^{-1}yv$

is in M. The factor that appears in the position occupied by x_{i_1} is $h_n^{-1}k_{i_n}^{-1}h_nk_{i_1}$. The element v was arbitrary so it may now be chosen so that it has factors in the positions corresponding to x_{i_1} and x_{i_n} such that the factor above is any element of H. But $y^{-1}v^{-1}yv$ is in M and is a multiplication, so it is also in N. This means that the group G is H for N = M ∩ V. We have shown:

<u>Theorem 5</u>: Let M be a normal subgroup of $\Sigma(H; B, d, d)$ not contained in the basis group. The multiplications contained in M form a normal subgroup N of $\Sigma(H; B, d, d)$ in which G = H. i.e., the factors in any fixed position run through the whole group and H/G_1 for N is an Abelian group.

Let us turn now to the permutations in M. Denote by P the intersection of M with S; P = M ∩ S. P is a subgroup of M and M is normal in Σ, so it follows that P is normal in S.

M is not contained in V, so there exists a substitution y of M containing a cycle c of length ≥ 2. We have seen in Chapter 2 that y is conjugate to an element y_0 containing a cycle $c_0 = \begin{pmatrix} x_1, & \dots, & x_n \\ x_2, & \dots, & ax_1 \end{pmatrix}$. Since M is normal in Σ, M must contain y_0. If $n \geq 3$ then we put $s = (1,2)$ and M must contain the commutator $y_0^{-1}sy_0s^{-1} = (1,2,3)$. If every substitution of M has cycles of lengths 1 or 2, since M was not contained in V, M must contain at least one element y with one cycle c of length 2 in its decomposition. Then y is conjugate to an element $y_\circ$ which contains a cycle $c_\circ = \begin{pmatrix} x_1, & x_2 \\ x_2, & ax_1 \end{pmatrix}$. Since M is normal in Σ, M must contain $y_\circ$. Every element of Σ maps an infinite number of x into themselves with factors of e. Let x_3 and x_4 be two of these x's that $y_\circ$ sends into themselves with factor e, and let $s = (1,3,4)$. Then M must contain the commutator $y_\circ^{-1} s y_\circ s^{-1} = (1,4,2)$. We have proved the following:

<u>Theorem 6</u>: Every normal subgroup M of $\Sigma(H;\ B,\ d,\ d)$ not contained in the basis group contains permutations.

Since the group P is normal in S(B,d), it is either S(B,d) or A(B,d) by Theorem 4 of Chapter I.

Ore [1] proved that for the group $\Sigma(H;\ n,\ n+1,\ n+1)$ every element of the subgroup $N = V \cap M$ has the property that the product of the factors is contained in G_1. We note that this result for the elements of the subgroup $N = V \cap M$ of $\Sigma(H;\ B,\ d,\ d)$ has already been attained in the preceding theorems.

We shall proceed to the actual construction of the normal subgroups of $\Sigma(H;\ B,\ d,\ d)$. The quotient group Σ/V is isomorphic to S. Since $M \cup V$ is the union of two normal subgroups of Σ, it is normal in Σ and $\frac{M \cup V}{V} \cong \frac{M}{N}$. This means that M/N is isomorphic to S(B,d) or A(B,d). P was also one of these groups.

Three possibilities arise. It may happen that (i) $M/N \cong S(B,d)$ and $P = S(B,d)$, (ii) $M/N \cong A(B,d)$ and $P = A(B,d)$, or (iii) $M/N \cong S(B,d)$ and $P = A(B,d)$.

Let us discuss first the case where $P \cong M/N$ and hence $M = N \cup P$. We shall prove:

<u>Theorem 7</u>: Let P be a normal subgroup of $S(B,d)$ and let N be a normal subgroup of the type discussed in Theorem 5. Then $M = N \cup P$ is a normal subgroup of $\Sigma(H; B, d, d)$.

<u>Proof</u>: The group N is normal in Σ by assumption, and hence it is sufficient to show that an element in P is transformed into an element in M. Since P is normal in S, it is sufficient to show that a permutation s in P is transformed into an element of M by a multiplication $v = \{e, \ldots, e, h_{i_1}, e, \ldots, e, h_{i_n}, e, \ldots\}$. The commutator $s^{-1}v^{-1}sv$ is a multiplication. Although we cannot say what factors occur in which positions, we can say that the product of the factors is in G_1 since H/G_1 is Abelian. Therefore, the commutator is in N which implies that $v^{-1}sv$ is in M, and we have shown M is normal in Σ.

It remains to discuss the case where $\frac{M \cup V}{V} \cong \frac{M}{N} \cong S(B,d)$ and $P = A(B,d)$. Since $P \subset M$ it follows that $(V \cup A) \subset (M \cup V)$. But if every element y of M could be written as $y = vs = sv_1$ where s belongs to A and v, v_1 belong to V, then every element of $M \cup V$ could be written as $sv_1v_2 = sv_3$ and every coset of $\frac{M \cup V}{V}$ would be of the form sV where s belongs to A. Thus if every element of M has as its permutation part only elements from A we would have $\frac{M \cup V}{V} \cong A$ contrary to the present assumption about $\frac{M \cup V}{V}$. This shows that M must contain an element y of the form $y = vs$ where v belongs to V, s belongs to S, and s does not belong to A. Now s leaves an infinite number of x fixed. Let x_1 and x_2 be two of these. The permutation $s^{-1}(1,2)$ belongs to $A = P \subset M$. So $ys^{-1}(1,2) = vss^{-1}(1,2) = v(1,2)$ must belong to M by the fact that M is closed under multiplication. We have shown that M must contain an element y_1, which has the cycle $c = \begin{pmatrix} x_1, & x_2 \\ a_1x_2, & a_2x_1 \end{pmatrix}$ in its cyclic decomposition and which maps all other x into themselves with only a finite number of factors different from e. According to Theorem 5 the factors of any element of N may be taken arbitrarily in H except for one of them, so we may choose an element v_1 of M from N such that all the factors of $v(1,2)v_1$ are e except those of x_1 and x_2. The element $v(1,2)v_1$ may be transformed so that M must contain an element of the form $y = \begin{pmatrix} x_1, & x_2 \\ x_2, & ax_1 \end{pmatrix}$. Since $y^2 \in N$ we see that a^2 belongs to G_1. We can now prove:

<u>Theorem 8</u>: Let $M = N \cup A(B,d)$ be a normal subgroup of $\Sigma(H;\ B,\ d,\ d)$ defined by the procedure of Theorem 7 and let L

be the cyclic subgroup generated by $y = \begin{pmatrix} x_1, & x_2 \\ x_2, & ax_1 \end{pmatrix}$ where a^2 belongs to G_1. Then $M_1 = L \cup M$ is a normal subgroup of $\Sigma(H; B, d, d)$.

Proof: Since M is normal in Σ it is sufficient to show that $(vs)y(vs)^{-1}$ belongs to M_1 for all vs of Σ because y^2 belongs to N. Since $(vs)y(vs)^{-1} = v(sys^{-1})v^{-1}$, it is sufficient to show that sys^{-1} and vyv^{-1} belong to M_1.

We shall show first that if s is any element of S(B,d) and $y = \begin{pmatrix} x_1, & x_2 \\ x_2, & ax_1 \end{pmatrix}$ then sys^{-1} belongs to M_1. We can write $s = \begin{pmatrix} x_i, & x_j, & \cdots \\ x_1, & x_2, & \cdots \end{pmatrix}$ where $i \neq j$. Then $sys^{-1} = \begin{pmatrix} x_i, & x_j \\ x_j, & ax_i \end{pmatrix}$. We know that the group A(B,d) must contain an element of the form $s_1 = \begin{pmatrix} x_i, & x_j, & \cdots \\ x_1, & x_2, & \cdots \end{pmatrix}$ since the mapping of the x's not shown may always be chosen in such a way that s_1 belongs to A(B,d). Now consider $s_1ys_1^{-1} = \begin{pmatrix} x_i, & x_j \\ x_j, & ax_i \end{pmatrix}$. The element $s_1ys_1^{-1}$ is in $A \cup L \subset M_1$. But this is the same as sys^{-1}. Therefore, sys^{-1} belongs to M_1 for all s of S(B,d).

It remains to show that vyv^{-1} belongs to M_1 for any v of V. Let $v = \{h_1, h_2, \ldots, h_\varepsilon, \ldots\}$ where all but a finite number of the h are e. The commutator

$$y^{-1}vyv^{-1} = \{a^{-1}h_2ah_1^{-1},\ h_1h_2^{-1},\ e, \ldots, e, \ldots\}$$

belongs to $N \subset M \subset M_1$ if $a^{-1}h_2ah_1^{-1}h_1h_2^{-1} = a^{-1}h_2ah_2^{-1}$ belongs to G_1. But since H/G_1 is Abelian the desired result follows.

We have established:

<u>Theorem 9</u>: The normal subgroups of $\Sigma(H;\ B,\ d,\ d)$ are given by Theorems $4'$, 7, and 8.

2. Normal Subgroups of $\Sigma_A(H;\ B,\ d,\ d)$

<u>Theorem 1</u>: The normal subgroups of $\Sigma_A(H;\ B,\ d,\ d)$ are precisely those normal subgroups of $\Sigma(H;\ B,\ d,\ d)$ which are contained in $\Sigma_A(H;\ B,\ d,\ d)$.

<u>Proof</u>: Let N be a normal subgroup of $\Sigma(H;\ B,\ d,\ d)$ contained in $\Sigma_A(H;\ B,\ d,\ d)$. Then N is certainly normal in $\Sigma_A(H;\ B,\ d,\ d)$.

Conversely, let N be a normal subgroup of $\Sigma_A(H;\ B,\ d,\ d)$. We claim that N is also normal in $\Sigma(H;\ B,\ d,\ d)$. If N is not normal in $\Sigma(H;\ B,\ d,\ d)$, then there exists an element y of N such that $(1,2)y(1,2)$ does not belong to N. The element $(1,2)y(1,2)$, like all elements of $\Sigma(H;\ B,\ d,\ d)$, maps an infinite number of x into themselves with e as a factor. Let x_α, x_β be two of these with the additional property that 1, 2, α, β are distinct. Then the element

$$(\alpha,\beta)(1,2)y(1,2)(\alpha,\beta) = (1,2)y(1,2)$$

will not belong to N. But this contradicts the normality of N in $\Sigma_A(H;\ B,\ d,\ d)$ since $(\alpha,\beta)(1,2)$ belongs to $A(B,d)$.

As a result of this theorem we see that the normal subgroups of $\Sigma_A(H;\ B,\ d,\ d)$ contained in the basis group are those described by Theorem $4'$ of Section 1. The normal subgroups of $\Sigma_A(H;\ B,\ d,\ d)$ not contained in the basis group are the subgroups described in Theorem 7 intersected with $\Sigma_A(H;\ B,\ d,\ d)$. That is, the normal subgroups of $\Sigma_A(H;\ B,\ d,\ d)$ are the union of groups N, as described by Theorem 5 of Section 1, and $A(B,d)$.

We have established:

Theorem 2: The normal subgroups of $\Sigma_A(H;\ B,\ d,\ d)$ are those described by Theorem $4'$ of Section 1 and the union of a group N, as described by Theorem 5 of Section 1, and $A(B,d)$.

3. Normal Subgroups of $\Sigma_A(H;\ n,\ n+1,\ n+1)$ for $n \geq 5$

We shall find all normal subgroups of $\Sigma_A(H;\ n,\ n+1,\ n+1)$, for $n \geq 5$, using the methods of Section 1 of this chapter.

3.1 Permutation Invariant Subgroups of $\Sigma_A(H;\ n,\ n+1,\ n+1)$ Contained in the Basis Group

Let N be a fixed permutation invariant subgroup of $\Sigma_A(H;\ n,\ n+1,\ n+1)$ contained in the basis group. We see, in the same way as in Section 1, that the $i^{\underline{th}}$ factors of elements in N run through a subgroup of H which is the same for all i because N is permutation invariant. We denote it by G. Let S_1 be the group of all elements of N for which $h_i = e$ for $i > 1$. S_1 is normal in N. The first factors of elements of S_1 run through a normal subgroup G_1 of G. Let $S^{(i)}$, $i = 2, \ldots, n$, be the set of elements sS_1s^{-1} where $s = (1,i)(j,k)$. The 1, i, j, k may and will be chosen such that they are distinct since $n \geq 5$. $S^{(i)}$ is a normal subgroup of N. Furthermore, $S^{(i)} \cap S^{(j)} = E$ for $i \neq j$. We denote by W the direct product $S_1 \times \prod_i S^{(i)}$, $i = 2, \ldots, n$. W is normal in N. Therefore, the relations between the factors of elements of N can only be determined modulo G_1. So we consider the quotient group G/G_1 and <u>assume $G_1 = e$</u> for the proofs of Theorems 1, 2, 3 of this section and Theorem 4 of Section 3.2.

Let S_2 be the group of those elements of N such that $h_i = e$ for $i > 2$. S_2 is normal in N. Let G_2 be the group that the first factors of elements of S_2 run through. Since S_2 is permutation invariant, the second factors of elements of S_2 also run through G_2. G_2 is normal in G. We may now apply the result of Ore [1; Theorem 1, p. 29] and conclude that elements of S_2 have the form

$$v = \{g_2, g_2^T, e, \ldots, e\}$$

where T is some automorphism of order two of the group G_2.

We shall now show $g_2^T = g_2^{-1}$. N is permutation invariant, so if it contains v as above it must contain $v_1 = svs^{-1} = \{g_2, e, g_2^T, e, \ldots, e\}$ where $s = (2,4,3)$.

Furthermore, N must contain $v_2 = v_1^{-1}v =$ $\{e, g_2^T, (g_2^T)^{-1}, e, \ldots, e\}$. Finally, N must contain $(1,2,3)v_2(1,3,2) = \{g_2^T, (g_2^T)^{-1}, e, \ldots, e\}$. Therefore, the automorphism T is such that $g_2^T = g_2^{-1}$. This means the group G_2 is Abelian. We have shown the following:

<u>Theorem 1</u>: Let N be a fixed permutation invariant subgroup of $\Sigma_A(H;\ n, n+1, n+1)$, for $n \geq 5$, contained in the basis group. Then the set G of H, consisting of all the factors that occur in a fixed $i^{\underline{th}}$ position of all multiplications of N form a normal subgroup of H. This group is the same for all i. The set S_1 of all multiplications of N which have $h_i = e$, for $i > 1$, form a normal subgroup of N. The set G_1, consisting of all first factors of multiplications of S_1 form a normal subgroup of G. We shall assume $G_1 = e$. The set S_2 of elements of N that have $h_i = e$, for $i > 2$, form a normal subgroup of N. The set G_2 of first factors of elements of S_2 form a normal Abelian subgroup of G. The elements of S_2 are of the form

$$v = \{g_2, g_2^{-1}, e, \ldots, e\}$$

where g_2 runs through G_2.

Let R be the subgroup of N generated by the substitutions obtained by all possible permutations of elements of S_2. In the same way as in Section 1.1 we can establish:

<u>Theorem 2</u>: The group R, generated by the substitutions obtained by all possible permutations of elements of S_2, consists of elements of the form

$$v = \{r_1, r_2, \ldots, (r_1 \ldots r_{n-1})^{-1}\}$$

where the r_i run through the Abelian group G_2 independently.

We now turn to the final step in the determination of the permutation invariant subgroups of $\Sigma_A(H;\ n,\ n+1, n+1)$ contained in the basis group. The nature of these groups is that of the permutation invariant subgroups of $\Sigma(H;\ n,\ n+1,\ n+1)$ contained in the basis group found by Ore [1]. Let $v = \{h_1, \ldots, h_n\}$ be an arbitrary element of N. Let $s_i = (1,i,j)$ where i runs over the set $2, \ldots, n$ and j is chosen such that it is different from both 1 and i. Since s_i is an element of $A(n,\ n+1)$, the element $s_i v s_i^{-1}$ must belong to N. $v_i = s_i v s_i^{-1} =$ $\{h_i, \ldots, h_{i-1}, h_j, h_{i+1}, \ldots, h_{j-1}, h_1, h_{j+1}, \ldots, h_n\}$ where for convenience we have let $i < j$, for $i = 2,\ldots,n$. Consider the element vv_i^{-1} which has the form $\overset{\circ}{v}_i = vv_i^{-1} = \{h_1h_i^{-1},\ldots,\ e,\ h_ih_j^{-1},\ e,\ldots,\ e, h_jh_1^{-1}, e,\ldots, e\}$. By hypothesis $n \geq 5$ so there exist natural numbers k, m each different from $1,i,j$. Consider $v_i' = (1,k,m)\overset{\circ}{v}_i(1,m,k)$ $= \{e,\ldots,e,\ldots,e,h_1h_i^{-1},e,\ldots,e,h_ih_j^{-1},e,\ldots,e,h_jh_1^{-1},e,\ldots,e\}$ where $h_1h_i^{-1}$ appears in the $m^{\underline{th}}$ position, $h_ih_j^{-1}$ in the $i^{\underline{th}}$ position, and $h_jh_1^{-1}$ in the $j^{\underline{th}}$ position. Form the element $\overset{\circ}{v}_i(v_i')^{-1} = \{h_1h_i^{-1},\ \ldots,\ e,\ h_ih_1^{-1},\ e,\ \ldots,\ e\}$ where all factors except those in the first and $m^{\underline{th}}$ position are e. This shows $h_1h_i^{-1}$ belongs to G_2 for $i = 2,\ldots,n$. Therefore, the elements of N must have the form $v = \{r_1g, r_2g, \ldots, r_ng\}$ where the r_i, $i = 1, \ldots, n$, are elements of G_2 and g runs through G.

It remains to determine the relations between the r_i in the above which will insure that the elements will form a permutation invariant group. The method used by Ore [1, p. 31] can be used and we reproduce it here.

Since R contains $v_1 = \{r_1, r_2, \ldots,(r_1\ldots r_{n-1})^{-1}\}$, when N contains $v_2 = \{r_1g, r_2g, \ldots, r_ng\}$, it must contain $v_1^{-1}v_2 = \{g, \ldots, g, g_\circ\}$. All elements of N of this form, form a group, and, since we are assuming $G_1 = e$, the factor $g_\circ$ is uniquely determined by g. This correspondence must be a homomorphism, and we can write $g\theta = g_\circ$ where θ is an endomorphism of G. When this is applied to $v_1 = \{r_1, r_2, \ldots, (r_1\ldots r_{n-1})^{-1}\}$, we have the final form for the elements of N is

$$v_2 = \{r_1g, r_2g, \ldots, r_{n-1}g, (r_1\ldots r_{n-1})^{-1}(g\theta)\}.$$

In $v_1^{-1}v_2$ we had $g\theta = g_\circ = rg$ where r is an element of G_2, and hence the endomorphism θ of G must transform an element g into an element obtained by multiplication of g by an element of G_2. When the special element $\{r, r, \ldots, r, r^{-(n-1)}\}$ in R is considered, it follows that $(r)\theta = r^{-(n-1)}$ for any element r of G_2. This relation shows that the last factor in v_2 is uniquely determined by the first n-1 factors.

For let

$$v_2 = \{r_1g, r_2g, \ldots, r_{n-1}g, (r_1\ldots r_{n-1})^{-1}(g\theta)\}.$$

If there exists an element in N of the form

$$v_3 = \{r_1rr^{-1}g, r_2rr^{-1}g, \ldots,$$
$$r_{n-1}rr^{-1}g, (r_1rr_2r\ldots r_{n-1}r)^{-1}((r^{-1}g)\theta)\},$$

then

$$(r_1r_2\ldots r_{n-1}r^{n-1})^{-1}((r^{-1}g)\theta) =$$
$$(r_1 \ldots r_{n-1})^{-1}r^{-(n-1)}(r^{-1}\theta)(g\theta) =$$
$$(r_1 \ldots r_{n-1})^{-1}r^{-(n-1)}(r\theta)^{-1}(g\theta) =$$
$$(r_1\ldots r_{n-1})^{-1}r^{-(n-1)}(r^{-(n-1)})^{-1}g\theta = (r_1\ldots r_{n-1})^{-1}(g\theta).$$

Conversely, under the stated conditions on θ and under the assumption G_2 is an Abelian group the elements of the form $\{r_1g, r_2g, \ldots, r_{n-1}g, (r_1\ldots r_{n-1})^{-1}(g\theta)\}$ form a group.

We shall show first that if

$$v = \{r_1g, r_2g, \ldots, r_{n-1}g, (r_1\ldots r_{n-1})^{-1}(g\theta)\}$$

is an element of the type under discussion that

$$v^{-1} = \{(r_1g)^{-1}, (r_2g)^{-1}, \ldots, (r_{n-1}g)^{-1}, (g\theta)^{-1}$$
$$(r_1 \ldots r_{n-1})\}$$

is of the same type. We can rewrite v^{-1} as

$$v^{-1} = \{g^{-1}r_1^{-1}, g^{-1}r_2^{-1}, \ldots, g^{-1}r_{n-1}^{-1}, (g\theta)^{-1}(r_1\ldots r_{n-1})\}$$
$$= \{\mathring{r}_1g^{-1}, \mathring{r}_2g^{-1}, \ldots, \mathring{r}_{n-1}g^{-1}, (g\theta)^{-1}(r_1\ldots r_{n-1})\}$$

where $\mathring{r}_i = g^{-1}r_i^{-1}g$. Then $(\mathring{r}_1\ldots\mathring{r}_{n-1})^{-1}(g^{-1}\theta) =$
$g^{-1}r_1gg^{-1}r_2g\ldots g^{-1}r_{n-1}g(g\theta)^{-1} = g^{-1}(r_1\ldots r_{n-1})g(g\theta)^{-1}.$

It is sufficient to show that

$$g^{-1}(r_1\ldots r_{n-1})g(g\theta)^{-1} = (g\theta)^{-1}(r_1\ldots r_{n-1}).$$

The endomorphism θ was such that $(g\theta) = rg$ for some r of G_2. Therefore, $g^{-1}(r_1\ldots r_{n-1})g(g\theta)^{-1} =$ $g^{-1}(r_1\ldots r_{n-1})g(rg)^{-1} = g^{-1}(r_1\ldots r_{n-1})r^{-1}$. On the other hand, we have $(g\theta)^{-1}(r_1\ldots r_{n-1}) = (rg)^{-1}(r_1\ldots r_{n-1})$ $= g^{-1}r^{-1}(r_1\ldots r_{n-1})$. Since G_2 is Abelian this concludes showing that v^{-1} is of the desired type.

Now let the two following elements

$$v = \{r_1g,\ r_1g,\ldots,r_{n-1}g,(r_1\ldots r_{n-1})^{-1}(g\theta)\},$$

$$v_1 = \{\overset{\circ}{r}_1\overset{\circ}{g},\ \overset{\circ}{r}_2\overset{\circ}{g},\ldots,\ \overset{\circ}{r}_{n-1}\overset{\circ}{g},(\overset{\circ}{r}_1\ldots\overset{\circ}{r}_{n-1})^{-1}(\overset{\circ}{g}\theta)\}$$

be given. We shall show that the element

$$vv_1 = \{r_1g\overset{\circ}{r}_1\overset{\circ}{g},\ldots,r_{n-1}g\overset{\circ}{r}_{n-1}\overset{\circ}{g},(r_1\ldots r_{n-1})^{-1}(g\theta)(\overset{\circ}{r}_1\ldots\overset{\circ}{r}_{n-1})^{-1}(\overset{\circ}{g}\theta)\}$$

is of the same form as v and v_1. The group G_2 is normal in G. We can, therefore, rewrite vv_1 as follows:

$$vv_1 = \{r_1r_1'g\overset{\circ}{g},\ldots,r_{n-1}r_{n-1}'g\overset{\circ}{g},(r_1\ldots r_{n-1})^{-1}(g\theta)(\overset{\circ}{r}_1\ldots\overset{\circ}{r}_{n-1})^{-1}(\overset{\circ}{g}\theta)\}.$$

This is accomplished by setting $g\overset{\circ}{r}_ig^{-1} = r_i'$ for $i = 1,\ \ldots,\ n-1$. We must now show that

$$(r_1r_1'\ \ldots\ r_{n-1}r_{n-1}')^{-1}((g\overset{\circ}{g})\theta) =$$
$$(r_1\ \ldots\ r_{n-1})^{-1}(g\theta)(\overset{\circ}{r}_1\ \ldots\ \overset{\circ}{r}_{n-1})^{-1}(\overset{\circ}{g}\theta).$$

Since G_2 is Abelian this reduces to showing that

$$((r_1' \ldots r_{n-1}')(r_1 \ldots r_{n-1}))^{-1}(ge)(g^\circ e) =$$

$$(r_1 \ldots r_{n-1})^{-1}(ge)(r_1^\circ \ldots r_{n-1}^\circ)^{-1}(g^\circ e).$$

It is, therefore, sufficient to show that

$$(ge)^{-1}(r_1' \ldots r_{n-1}')^{-1}(ge) = (r_1^\circ \ldots r_{n-1}^\circ)^{-1}.$$

The endomorphism e was such that $ge = rg$ where r is an element of G_2. Therefore,

$$(ge)^{-1}(r_1' \ldots r_{n-1}')^{-1}(ge) =$$

$$g^{-1}r^{-1}(gr_1^\circ g^{-1}gr_2^\circ g^{-1} \ldots gr_{n-1}^\circ g^{-1})^{-1}rg =$$

$$g^{-1}r^{-1}(gr_1^\circ r_2^\circ \ldots r_{n-1}^\circ g^{-1})^{-1}rg =$$

$$g^{-1}r^{-1}g(r_1^\circ \ldots r_{n-1}^\circ)^{-1}g^{-1}rg.$$

But using the fact that G_2 is normal in G we have $g^{-1}r^{-1}g = \bar{r}_1$ and $g^{-1}rg = \bar{r}_1^{-1}$. Since G_2 is Abelian

$$g^{-1}r^{-1}g(r_1^\circ \ldots r_{n-1}^\circ)^{-1}g^{-1}rg =$$

$$\bar{r}_1(r_1^\circ \ldots r_{n-1}^\circ)^{-1}\bar{r}_1^{-1} = (r_1^\circ \ldots r_{n-1}^\circ)^{-1}.$$

The group of elements of this form is permutation invariant. For let v be an element of the form

$$v = \{r_1g, r_2g, \ldots, r_{n-1}g, (r_1 \ldots r_{n-1})^{-1}(ge)\}.$$

If any two of the factors are permuted to give an element v_1, the element vv_1^{-1} is in the group. Let us illustrate with a particular element:

$$v_1 = \{r_2g, r_1g, \ldots, r_{n-1}g, (r_1 \ldots r_{n-1})^{-1}(ge)\},$$

$$vv_1^{-1} = \{r_1gg^{-1}r_2^{-1}, r_2gg^{-1}r_1^{-1}, e, \ldots, e\},$$

$$vv_1^{-1} = \{r_1r_2^{-1}, r_2r_1^{-1}, e, \ldots, e\}.$$

This is clearly an element of the desired type where $g = e$, $r_i = e$ for $i > 2$. The last factor is as it should be since $(r_1 r_2^{-1} r_2 r_1^{-1})^{-1}(g\theta) = e$. It is clear that if any two of the first n-1 factors are permuted the result is similar to the illustrated one and belongs to the set of elements under discussion.

Let us now consider the case where one of the two factors permuted is the $n^{\underline{th}}$ one. This means that v_1 has the form:

$$v_1 = \{(r_1 \ldots r_{n-1})^{-1}(g\theta),\ r_2 g, \ldots, r_{n-1} g,\ r_1 g\}.$$

$$vv_1^{-1} = \{r_1 g(g\theta)^{-1}(r_1 \ldots r_{n-1}),\ e,\ \ldots,\ e,\ (r_1 \ldots r_{n-1})^{-1}(g\theta)g^{-1}r_1^{-1}\}.$$

The endomorphism θ sends g into rg for some r of G_2. Therefore,

$$r_1 g(g\theta)^{-1}(r_1 \ldots r_{n-1}) =$$

$$r_1 g(rg)^{-1}(r_1 \ldots r_{n-1}) = r_1 r^{-1} r_1 \ldots r_{n-1}.$$

The element vv_1^{-1} would be of the desired type if

$$(r_1 \ldots r_{n-1})^{-1}(g\theta)g^{-1}r_1^{-1} = (r_1 r^{-1} r_1 \ldots r_{n-1})^{-1}.$$

But

$$(r_1 \ldots r_{n-1})^{-1}(g\theta)g^{-1}r_1^{-1} = (r_1 \ldots r_{n-1})^{-1} rgg^{-1}r_1^{-1} =$$

$$(r_1 \ldots r_{n-1})^{-1} rr_1^{-1} = (r_1 r^{-1} r_1 \ldots r_{n-1})^{-1}$$

since $(g\theta) = rg$ and G_2 is Abelian.

We have established:

<u>Theorem 3</u>: Let a subgroup G of H be chosen. In G a normal Abelian subgroup G_2 is chosen. Then the group N, consisting of all elements of the form

$$v = \{r_1 g, r_2 g, \ldots, r_{n-1} g, (r_1 \ldots r_{n-1})^{-1}(g\theta)\},$$

where

(1) the r_i are arbitrary elements of G_2,

(2) g runs through G,

(3) θ is an endomorphism of G multiplying each element of G by an element of G_2, and

(4) in particular, $(g)\theta = g^{-(n-1)}$ for g belonging to G_2,

is a permutation invariant subgroup of $\Sigma_A(H; n, n+1, n+1)$ for $n \geq 5$.

We recall that for convenience the subgroup G_1 was assumed to be e. The results obtained may, therefore, be generalized by working with G_1, a normal subgroup of G, and $G_1 \neq e$. Such a consideration leads to a determination of all permutation invariant subgroups of $\Sigma_A(H; n, n+1, n+1)$ contained in the basis group. This result is stated in the following theorem:

<u>Theorem 3'</u>: All permutation invariant subgroups N of $\Sigma_A(H; n, n+1, n+1)$, for $n \geq 5$, that are contained in the basis

group may be obtained by the following construction. A subgroup G of H is chosen. In G two normal subgroups $G_1 \subset G_2$ are selected such that the quotient group G_2/G_1 is Abelian. Then let N consist of elements of the form

$$v = \{k_1, k_2, \ldots, k_n\}$$

where the k_i runs through G subject to the conditions

$$\left.\begin{array}{l} (1)\ k_i \equiv r_i k,\ i = 1, \ldots, n-1 \\ (2)\ k_n \equiv (r_1 \ldots r_{n-1})^{-1}(k\theta) \end{array}\right\} \bmod G_1$$

where the r_i are arbitrary elements of G_2. Furthermore, θ is an endomorphism of G/G_1 multiplying each element of G/G_1 by an element of G_2/G_1. In particular, $(g)\theta \equiv g^{-(n-1)} \bmod G_1$ for any element of G_2.

3.2 Normal Subgroups of $\Sigma_A(H; n, n+1, n+1)$ Contained in the Basis Group

We shall have to find those normal subgroups of the basis group which are permutation invariant. We shall use the same notation as in the previous section. Let N be a normal subgroup of Σ contained in $V(n, n+1)$. The groups $G_1 \subset G_2 \subset G$ are normal in H. Since N is permutation invariant, an element v of N must have the form described in Theorem $3'$. For convenience we assume $G_1 = e$. Then any factor of an element of N is uniquely determined by the other factors. Let $v_1 = \{h, e, \ldots, e\}$ be an element of the basis group that has only one non-identity factor and it occurs in the first position. The element h is arbitrary in H. Let $v = \{r_1 g, r_2 g, \ldots, (r_1 \ldots r_{n-1})^{-1}(ge)\}$ be any element of N. Then $v_1 v v_1^{-1} = \{hr_1gh^{-1}, r_2g, \ldots, (r_1 \ldots h_{n-1})^{-1}(ge)\}$. This shows that $hr_1gh^{-1} = r_1g$. Therefore, r_1g belongs to the center of H. We have shown:

<u>Theorem 4</u>: Let N be a normal subgroup of $\Sigma_A(H; n, n+1, n+1)$, for $n \geq 5$, contained in the basis group. Then N is permutation invariant and must meet the requirements of Theorem $3'$. Assume $G_1 = e$. The groups $G_2 \subset G$ are normal in H and G belongs to the center of H. Conversely, if N is as given by Theorem 3 and the additional requirements that $G_2 \subset G$ are normal in H and G belongs to the center of H are met, then N is normal in $\Sigma_A(H; n, n+1, n+1)$, for $n \geq 5$.

The assumption $G_1 = e$ was not necessary and we now generalize Theorem 4 by assuming $G_1 \neq e$. The factors of elements of N are uniquely determined modulo G_1. Then $hr_1gh^{-1} \equiv r_1g$ modulo G_1 and G/G_1 belongs to the center of H/G_1.

If $G_1 \subset G_2 \subset G$ are normal subgroups of H, G/G_1 belongs to the center of H/G_1, and N is permutation invariant, then N is normal in $\Sigma_A(H;\ n,\ n+1,\ n+1)$, for $n \geq 5$. This establishes:

<u>Theorem 4'</u>: For $n \geq 5$ the normal subgroups of $\Sigma_A(H;\ n,\ n+1,\ n+1)$ are obtained by the construction of Theorem 3' with the additional conditions

(1) $G_1 \subset G_2 \subset G$ are normal in H,

(2) G/G_1 belongs to the center of H/G_1.

3.3 <u>Other Normal Subgroups of $\Sigma_A(H;\ n,\ n+1,\ n+1)$</u>

By the method used to prove Theorem 5 of Section 1.3 we can prove:

<u>Theorem 5</u>: Let M be a normal subgroup of $\Sigma_A(H;\ n,\ n+1,\ n+1)$ not contained in the basis group. The multiplications $N = M \cap V$ form a normal subgroup of $\Sigma_A(H;\ n,\ n+1,\ n+1)$ in which $H = G$. i.e., the factors in any fixed position run through the whole group H and the quotient group H/G_1 for N is an Abelian group.

Let P be the subgroup of M consisting of permutations only; $P = M \cap A$. Since M is normal in Σ, it follows that P is normal in A. Hence, $P = A(n,\ n+1)$ or P is the identity. We now prove $P = A$.

Theorem 6: Every normal subgroup M of $\Sigma_A(H;\ n, n+1, n+1)$ not contained in the basis group contains permutations.

Proof: Since M is not contained in the basis group, there exists an element $y = vs$ where $s \neq I$. It is convenient to consider several cases.

Case 1. If y contains a cycle c in its cyclic decomposition of length $n \geq 4$, we have seen in Chapter 2 that y is conjugate to an element y_0 containing a cycle $c_0 = \begin{pmatrix} x_1, & x_2, & x_3, & \dots, & x_n \\ x_2, & x_3, & x_4, & \dots, & ax_1 \end{pmatrix}$. Since M is normal in Σ, M must contain y_0. Let $s = (1,2,3)$ and M must contain $y_0^{-1}sy_0s^{-1} = (1,3,4)$.

Case 2. If y contains a cycle c of length 3 and some other cycle of length greater than 1, then M contains an element y_0 conjugate to y of the form $y_0 = \begin{pmatrix} x_1, & x_2, & x_3 \\ x_2, & x_3, & ax_1 \end{pmatrix}\begin{pmatrix} x_4, & \dots \\ x_5, & \dots \end{pmatrix}\dots$. Let $s = (1,4,2)$ and M must contain $y_0^{-1}sy_0s^{-1} = (1,2,5,3\dots)$ which reduces to case 1.

It may happen that y contains one 3-cycle and the remainder are 1-cycles. We discuss this case later.

Case 3. If y contains only 1 or 2 cycles in its cyclic decomposition and has four 2-cycles, then M must contain an element y_0 conjugate to y of the form

$$y_0 = \begin{pmatrix} x_1, & x_2 \\ x_2, & ax_1 \end{pmatrix}\begin{pmatrix} x_3, & x_4 \\ x_4, & bx_3 \end{pmatrix}\begin{pmatrix} x_5, & x_6 \\ x_6, & cx_5 \end{pmatrix}\begin{pmatrix} x_7, & x_8 \\ x_8, & dx_7 \end{pmatrix} \dots .$$

Let $s = (1,3)(5,7)$. Then M must contain

$y_\circ^{-1}s^{-1}y_\circ s = (1,3)(2,4)(5,7)(6,8)$.

Case 4. We may now assume that y contains at the most two 2-cycles in its cyclic decomposition. M contains an element $y_\circ$ conjugate to y of the form

$$y_\circ = \begin{pmatrix} x_1, & x_2 \\ x_2, & ax_1 \end{pmatrix}\begin{pmatrix} x_3, & x_4 \\ x_4, & bx_3 \end{pmatrix}\begin{pmatrix} x_5, & \cdots \\ cx_5, & \cdots \end{pmatrix}.$$

Let $s = (1,3,5)$. Then M must contain

$y_\circ^{-1}sy_\circ s^{-1} = \begin{pmatrix} x_1, & x_2, & x_3, & x_4, & x_5 \\ x_5, & x_4, & x_1, & cx_3, & c^{-1}x_2 \end{pmatrix}$ which reduces to case 1.

Case 5. If y has one 3-cycle and the remainder 1-cycles, then M must contain an element $y_\circ$ conjugate to y of the form $y_\circ = \begin{pmatrix} x_1, & x_2, & x_3 \\ x_2, & x_3, & ax_1 \end{pmatrix}\begin{pmatrix} x_4, & x_5, & \cdots \\ bx_4, & cx_5, & \cdots \end{pmatrix}$. Let $s = (1,2,4)$. Then M must contain

$y_\circ^{-1}sy_\circ s^{-1} = \begin{pmatrix} x_1, & x_2, & x_3, & x_4 \\ x_4, & x_3, & bx_2, & b^{-1}x_1 \end{pmatrix}$ which reduces to case 4.

This concludes the proof of Theorem 6.

<u>Theorem 7</u>: The elements of the group $N = M \cap V$, where M is a normal subgroup of $\Sigma_A(H;\, n,\, n+1,\, n+1)$ and M is not contained in the basis group, are of the form

$$v = \{h_1, \ldots, h_n\}$$

where the h_i runs through H subject to the condition $h_1 \ldots h_n$ belongs to G_1. Here G_1 is a normal subgroup of H such that H/G_1 is Abelian.

<u>Proof</u>: It follows from Theorem 6 that M contains $s = (1,2,3)$. Let $v = \{h_1, \ldots, h_n\}$ be an arbitrary element of the basis group. Then M must contain $s^{-1}v^{-1}sv = \{h_3^{-1}h_1, h_1^{-1}h_2, h_2^{-1}h_3, e, \ldots, e\}$. Choose $h_2 = h_3$ and let them be fixed in H. Then $h_3^{-1}h_1$ can be made any element h of H by a proper choice of h_1, namely h_3h. Then $h_1^{-1}h_2 = h^{-1}h_3^{-1}h_2 = h^{-1}h_3^{-1}h_3 = h^{-1}$ and M contains $\{h, h^{-1}, e, \ldots, e\}$. This shows that $G_2 = H$.

For the moment let us again assume $G_1 = e$. Then $G_2 = H$ is Abelian. It follows from Theorem 3 that the elements of N must have the form

$$v = \{r_1g, \ldots, r_{n-1}g, (r_1 \ldots r_{n-1})^{-1}(g\theta)\}.$$

But θ is an endomorphism sending elements g of G_2 into $g^{-(n-1)}$. Therefore, $r_1g \ldots r_{n-1}g(r_1 \ldots r_{n-1})^{-1}(g\theta) = r_1g \ldots r_{n-1}g(r_1 \ldots r_{n-1})^{-1}g^{-(n-1)} = e$.

If $G_1 \neq e$, then H/G_1 is Abelian. It follows from Theorem 6 that factors of elements of N are as described above except they may be multiplied by elements from G_1. When the product of the factors from a typical multiplication of N is computed, since H/G_1 is Abelian, it is found to be an element of G_1.

The method of proof used here is that used by Ore [1, p. 36].

We shall now proceed to the actual construction of the normal subgroups of $\Sigma_A(H; n, n+1, n+1)$. The quotient group Σ/V is isomorphic to A. Furthermore, M/N is isomorphic to A. We now prove:

<u>Theorem 8</u>: Let N be a normal subgroup of $\Sigma_A(H;\ n,\ n+1,\ n+1)$ contained in the basis group of the type described in Theorem 7. Then $M = N \cup A(n,\ n+1)$ is a normal subgroup of $\Sigma_A(H;\ n,\ n+1,\ n+1)$. Conversely, if M is a normal subgroup of $\Sigma_A(H;\ n,\ n+1,\ n+1)$ not contained in the basis group then $M = N \cup A(n,\ n+1)$, where N is of the type described in Theorem 7.

<u>Proof</u>: N is normal in Σ by assumption, and hence it is only necessary to show an element of A is transformed into an element of M. Furthermore, it is only necessary to show that an element s of A is transformed into an element of M by a multiplication. Let $v = \{h_1,\ \ldots,\ h_n\}$ be any element of V. The commutator $v^{-1}svs^{-1} = \{h_1^{-1}h_{i_1},\ \ldots,\ h_n^{-1}h_{i_n}\}$ belongs to V. Again we shall discuss two situations. If $G_1 = e$ and H is Abelian, the product of the factors of the commutator is e. Hence, the commutator belongs to $N \subset M$. If $G_1 \neq e$, then H/G_1 is Abelian and the product of the factors of the commutator is in G_1. The commutator is in $N \subset M$. In any case $v^{-1}svs^{-1} \varepsilon N \subset M$. Therefore, M is normal in Σ.

Conversely, if M is normal in Σ and is not contained in the basis group, then M contains $A(n,\ n+1)$ by

Theorem 6. Now let $y = vs$ be any element of M. The permutation s belongs to $A(n, n+1)$. Since $A(n, n+1) \subset M$, M must also contain $ys^{-1} = v$. This shows that the multiplication part of any element of M is in N as given by Theorem 7. It follows that $M = N \cup A(n, n+1)$.

4. Normal Subgroups of $\Sigma_A(H; 2, 3, 3)$

The group $\Sigma_A(H; 2, 3, 3)$ is, of course, simply $V(2,3)$ and the problem is to find all normal subgroups of $H \times H$. The procedure of the preceding sections shall be used.

Let N be a fixed normal subgroup of $V(2,3)$. The elements of H that occur as first factors of elements of N form a group we shall denote by G_1. The second factors of elements of N form a group, G_2 say. Since N is normal in V, G_1 and G_2 are normal in H.

The set of elements S_1 of N of the form $v = \{g_1, e\}$ form a normal subgroup of N. The first factors of elements of S_1 form a normal subgroup $\overset{\circ}{G}_1$ of G_1. Similarly, the set of elements S_2 of N of the form $v = \{e, g_2\}$ form a normal subgroup of N and the second factors run through a normal subgroup $\overset{\circ}{G}_2$ of G_2.

Let W be the direct product of S_1 and S_2. W is normal in N since $\overset{\circ}{G}_1$, $\overset{\circ}{G}_2$ are normal in G_1, G_2 respectively. If $v = \{h_1, h_2\}$ is any element of N, then h_1 can be multiplied by any element of $\overset{\circ}{G}_1$ and h_2 can be multiplied by any element of $\overset{\circ}{G}_2$ and the resulting multiplication is in N. The relations on the first factors of elements of N can only be determined modulo $\overset{\circ}{G}_1$, and the relations on the second factors can only be determined modulo $\overset{\circ}{G}_2$. We shall work with the quotient groups $G_1/\overset{\circ}{G}_1$ and $G_2/\overset{\circ}{G}_2$ and <u>assume $\overset{\circ}{G}_1 = \overset{\circ}{G}_2 = e$</u>.

If N contains the two elements $v_1 = \{g_1, g_2\}$ and $v_2 = \{g_1, \overset{\circ}{g}_2\}$, then N contains $v_1 v_2^{-1} = \{e, g_2(\overset{\circ}{g}_2)^{-1}\}$. We have assumed $\overset{\circ}{G}_2 = e$. Therefore, $g_2(\overset{\circ}{g}_2)^{-1} = e$, or $g_2 = \overset{\circ}{g}_2$. Conversely, if the second factors of two elements of N are the same, the first factors are the same. This establishes a correspondence θ between G_1 and G_2 defined by $g_2 = g_1\theta$ where $v = \{g_1, g_2\}$ is any element of N. This correspondence is an isomorphism.

Let N contain $v = \{g_1, g_1\theta\}$. Let $v_1 = \{h, e\}$ be an element of the basis group where h is arbitrary in H. Since N is normal by assumption, N must contain $v_1 v v_1^{-1} = \{hg_1h^{-1}, g_1\theta\}$. But $g_1\theta$ is uniquely determined by the first factor. Therefore, $hg_1h^{-1} = g_1$. This means that G_1 belongs to the center of H. In a similar fashion it can be shown that G_2 belongs to the center of H.

Conversely, if the following set of conditions

(1) $G_1 \overset{\theta}{\cong} G_2$,

(2) G_1 belongs to the center of H,

(3) G_2 belongs to the center of H

are satisfied, then the set of elements of the form $v = \{g_1, g_2\}$, when g_1 runs through G_1, g_2 runs through G_2 subject to $g_2 = g_1\theta$, form a normal subgroup of V.

We have used the fact that a subgroup in the center of H is normal in H. We have shown:

<u>Theorem 1</u>: Choose G_1 and G_2 as subgroups of the center of H such that $G_1 \overset{e}{\cong} G_2$. Then the set of elements of the form

$$v = \{g_1, g_2\},$$

where g_1 runs through G_1 and g_2 runs through G_2 subject to $g_2 = g_1 e$, form a normal subgroup of $\Sigma_A(H; 2, 3, 3)$.

The assumption that $\mathring{G}_1 = \mathring{G}_2 = e$ was not necessary. The result above may be generalized by working with groups $\mathring{G}_1$ and $\mathring{G}_2$ different from e. Such a consideration leads to a determination of all normal subgroups of $\Sigma_A(H; 2, 3, 3)$. The result is stated in the following theorem:

<u>Theorem 2</u>: All the normal subgroups N of $\Sigma_A(H; 2, 3, 3)$ are given by the following construction. In H choose groups $\mathring{G}_1 \subset G_1$, $\mathring{G}_2 \subset G_2$ such that $G_1/\mathring{G}_1 \overset{e}{\cong} G_2/\mathring{G}_2$. The groups must also be chosen subject to $G_1/\mathring{G}_1$ belongs to the center of $H/\mathring{G}_1$ and $G_2/\mathring{G}_2$ belongs to the center of $H/\mathring{G}_2$. Then let N consist of the elements of the form

$$v = \{g_1, g_2\}$$

where g_1 runs through G_1 and g_2 runs through G_2 subject to $g_2 \equiv g_1 e \mod \mathring{G}_2$.

CHAPTER V

The Basis Group as Characteristic Subgroup

As has been the case in all previous discussions, we shall make no assumptions about the order of the group H with respect to which the symmetry is constructed.

<u>Theorem 1</u>: The basis group of $\Sigma(H; B, d, d)$ is a characteristic subgroup of the symmetry.

<u>Proof</u>: Let us assume that the theorem is not true. Then there exists some automorphism θ such that $(V)\theta \not\subseteq V$. There also exists some normal subgroup M of Σ such that $M\theta = V$. But this means that $M = V\theta^{-1}$. We already have $V\theta \not\subseteq V$ so $V \not\subseteq V\theta^{-1} = M$.

The quotient group Σ/V is isomorphic to S. We assert that $\Sigma/M \cong S \cong \Sigma/V$. We shall exhibit an isomorphism between Σ/M and S. We first write Σ in terms of cosets of M. Then $(My)\theta = \{m\}\theta(y)\theta = \{v\}(y\theta) = V(y\theta) = Vvs = Vs$. Define a correspondence ϕ between cosets of M and S by $(My)\phi = s$ where s is defined as above. ϕ is the desired isomorphism.

Consider the two normal groups K and N of Σ given by $K = V \cup M$ and $N = V \cap M$. The quotient group K/M is a normal subgroup of Σ/M. Since $V \not\subseteq M$, the group K/M is not the identity. Hence, K/M is isomorphic to a normal subgroup of S, i.e. S or A. Therefore, in any case, K/M is non-Abelian. On the other hand, from the second isomorphism law it follows that $K/M \cong V/N$. The form of the group N has been determined in Theorem $4'$, Section 1.1, Chapter IV. A normal subgroup G_1 of H is chosen such that H/G_1 is Abelian. Then N consists of all multiplications v where all but a finite number of the factors are e and the non-identity factors run through H subject to $h_1 \dots h_n$ belongs to G_1. This shows that the quotient group V/N is isomorphic to H/G_1, hence is Abelian. This contradicts $K/M \cong V/N$ since K/M is non-Abelian.

<u>Theorem 2</u>: The basis group of $\Sigma_A(H; B, d, d)$ is a characteristic subgroup of the symmetry.

The proof of Theorem 2 is similar to that of Theorem 1.

<u>Theorem 3</u>: The basis group of $\Sigma_A(H; n, n+1, n+1)$ for $n \geq 5$ is a characteristic subgroup of the symmetry.

The proof of Theorem 3 is similar to that of Theorem 1.

BIBLIOGRAPHY

1. Ore, O., Theory of Monomial Groups, Trans. of Amer. Math. Society, Vol. 51, No. 1 (1942), pp. 15-64.

2. Scott, W. R., Infinite Symmetric Groups, unpublished manuscript.

3. Carmichael, R. D., Introduction to the Theory of Groups of Finite Order, Illinois, Ginn and Company, 1937.

4. Schreier, J. and Ulam S., Über die Automorphismen der Permutationsgruppe der Natürlichen Zahlenfolge, Fundamenta Math., vol. 28 (1937), pp. 258-260.

5. Baer, R., Die Kompositionsreihe der Gruppe aller Eineindeutigen Abbildungen einer Unendlichen Menge auf sich, Studia Math., Vol. 5 (1934), pp. 15-17.